Pricing swaps and options on quadratic variation under stochastic time change models - discrete observations case *

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Abstract

We use a forward characteristic function approach to price variance and volatility swaps and options on swaps. The swaps are defined via contingent claims whose payoffs depend on the terminal level of a discretely monitored version of the quadratic variation of some observable reference process. As such a process we consider a class of Lévy models with stochastic time change. Our analysis reveals a natural small parameter of the problem which allows a general asymptotic method to be developed in order to obtain a closed-form expression for the fair price of the above products. As examples, we consider the CIR clock change, general affine models of activity rates and the 3/2 power clock change, and give an analytical expression of the swap price. Comparison of the results obtained with a familiar log-contract approach is provided.

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1 Introduction

The modern theory of asset pricing requires a well-defined notion of stochastic integral in order to describe the gains from dynamic trading strategies conducted in continuous time. The widest class of processes for which stochastic integrals are defined are semi-martingales, which loosely speaking, arise as the sum of a bounded variation process and a local martingale. The quadratic variation of a semi-martingale is a continuous time process which loosely speaking, arises by integrating over time the squared increments of the semi-martingale. The financial markets have recently seen the introduction of contingent claims whose payoffs depend on the terminal level of a discretely monitored version of the quadratic variation of some observable reference process. When annualized, this random variable is frequently referred to as the realized variance. The most popular example of such a contract is a variance swap (an example of one is given in Appendix 1). Less liquid examples of such contracts are volatility swaps and options on realized variance.

The greater liquidity of variance swaps relative to other contracts on realized variance is probably due to the existence of a well-known strategy for replicating the terminal level of quadratic variation under idealized conditions. This strategy combines dynamic trading in the underlying with a static position in a strip of co-terminal options of all positive strikes. Theoretically, if one can observe the initial prices of all of these options, one can calculate the theoretical variance swap rate [1]. In reality, market quotes for variance swap rates account for both missing strikes and option market illiquidity. In fact, the illiquidity of deeper out-of-the-money options is usually so pronounced that most market makers avoid taking positions in them, thereby adding to the replication error induced by missing strikes. The resulting possibility of loss is typically then compensated for through the setting of the variance swap quote.

When replicating variance swaps in this manner, at least two sources of errors can occur in practice:

1. Interpolation/extrapolation error due to the finite number of available option quotes relative to the continuum of option quotes needed to create the log contract.

2. Errors due to third and higher order powers of daily returns, often due to jumps.

The absence of market option prices suggests the use of parametric models which are capable of achieving consistency with the observed option prices. Examples of such models include local and stochastic volatility models or combinations of both. The reality of jumps further suggests using more sophisticated jump-diffusion and pure jump models to price swaps and options on quadratic variation. Among multiple papers on the subject, note the following [2, 3, 4, 5]. One can also combine the use of stochastic volatility models and jump models by subjecting jump processes to stochastic time change. For instance, in [2] variance swaps were priced using a familiar log-contract approach by computing a characteristic function of some jump processes with stochastic time change. The latter could be introduced either by a known distribution of the stochastic time (as in a celebrated VG model of Madan and Seneta [6]) or by a given SDE which describes evolution of the stochastic time (as an example, Carr, Geman, Madan and Yor use as the rate of time change the well-know CIR process [7])

Monte-Carlo methods can be used to price the quadratic variation products within these models. Unfortunately, analytical and semi-analytical (eg. FFT) results are available only for the simplest versions of these models. For instance, Swishchuk [8] uses the change-of-time method for the Heston model to derive explicit formulas for variance and volatility swaps. Also, Carr et. al. [4] proposed a method of pricing options on quadratic variation in Lévy models via the Laplace transform.

In the present paper we consider a class of models that are known to be able to capture at least the average behavior of the implied volatilities of the stock price across moneyness and maturity - time-changed Lévy processes. We derive an analytical expression for the fair value of the quadratic variation and volatility swap contracts as well as use the approach similar to that of [9] to price options on these products.

Our main contribution made in this paper is:

- In contrast to Car and Lee [10] who investigated variance swaps under continuous observations here we consider variance and volatility swaps under discrete observations.
- We use a forward characteristic function approach and propose a new asymptotic method which allows an analytical representation for the quadratic variation of a Lévy process with stochastic time change, if the latter is an affine process, and the annualized time between the observations is relatively small[1].
- We consider the activity rate models with a rather general jump specification proposed by Carr and Wu [11]. Using our method we prove (Theorem 1) that under this specification the annualized quadratic variation of the Lévy process with stochastic time determined by a **pure diffusion process** is given by the annualized realized variance times a constant coefficient ξ. This coefficient is determined via derivatives of the characteristic function of the underlying Lévy process.
- We also prove (Theorem 2) that given the above conditions the annualized quadratic variation of the Lévy process under stochastic time determined by a **jump-diffusion process** is also given by a product of the annualized realized variance and a constant coefficient ξ **plus** some constant η which is determined via derivatives of the characteristic function of the underlying Lévy process and jump integrals of the time change process.
- We further extend our results by investigating a more general case when discrete observations of the underlying spot price occur over a bigger time interval. We show (Theorem 3) that in this case the formulae for the price of the quadratic variation swap acquire two extra terms. The first one $p_0(\tau)$ is a function of time between observations τ and is determined by a particular model of the underlying Lévy process. The last term $p_2(\tau)\mathbb{E}_{\mathbb{Q}}[V^2]$ is proportional to the square of variance and is some kind of convexity adjustment.
- In addition to this general results we derive an analytical representation of the variance and volatility swaps for some particular models, namely the Lévy processes with the CIR time change and so-called "3/2 power" time change. The latter model does not belong to the class of the affine models, therefore this result further extends the proposed approach.

The rest of the paper is organized as follows. In section 2 we define a forward characteristic function (FCF) and how the quadratic variation of some stochastic process is related to this function. In section 3 we give a general representation of FCF for the Lévy process with stochastic time change. Section 4 considers in details a particular example of the time change which follows a well-known CIR process. First we propose an asymptotic method which allows us to derive an analytical expression for the quadratic variation of such a process under an arbitrary Lévy model. As examples, Heston and stochastic skew model of Carr and Wu are considered. Then we are discussing volatility swaps and options on the quadratic variation and show how to price them analytically within the framework of the proposed approach. Section 5 generalizes these results for a wide class of the time-change processes that have an affine activity rate. A general theorem is proved which again provides an analytical representation of the quadratic variation of such a process. In section 6 we extend our approach to one more class of the stochastic time change processes which follows so-called "3/2 power" clock change. We show that despite this model is not affine it still allows variance swaps to be priced in a closed form. Based on the results obtained a comparison of various models with respect to modeling variance swaps is provided in section 7. We examine the Heston model (Black-Scholes with the CIR time change, SSM model and NIG model also with the CIR time change and discuss the results.

[1] A more precise definition is given in the body of the paper

2 Quadratic variation and forward characteristic function

Let S_t denote the observable price at time $t \geq 0$ of some reference index, which is assumed to be strictly positive. The discretely monitored quadratic variation of the stochastic process $s_t = \log S_t/S_0$ after $N+1$ observations is a random variable defined as follows:

$$\sum_{i=1}^{N} \left(s_{t_i} - s_{t_{i-1}}\right)^2 \tag{1}$$

The annualized version of this random variable is defined as:

$$\frac{k}{N}\sum_{i=1}^{N} \left(s_{t_i} - s_{t_{i-1}}\right)^2, \tag{2}$$

ni where k is the number of periods per year eg. 252.

Suppose that a variance swap matures at time T and further suppose that the observations are uniformly distributed over $(0,T)$ with $\tau = t_i - t_{i-1} = const, \forall i = 1, N$. For each dollar of notional, the floating part of the payoff on the variance swap is defined as

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{k}{N}\sum_{i=1}^{N} \left(s_{t_i} - s_{t_{i-1}}\right)^2\right]. \tag{3}$$

Like all swaps, the variance swap has zero cost of entry and the magnitude of the fixed payment is determined at inception. Assuming no arbitrage, there exists a probability measure $\mathbb{Q}$ such that the fixed payment per dollar of notional can be presented as:

$$\mathcal{Q}_N(s) \equiv \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{T}\sum_{i=1}^{N}(s_{t_i} - s_{t_{i-1}})^2\right] = \frac{1}{T}\sum_{i=1}^{N}\mathbb{E}_{\mathbb{Q}}\left[(s_{t_i} - s_{t_{i-1}})^2\right], \tag{4}$$

Note, that quadratic variation is often used as a measure of realized variance. Moreover, modern variance and volatility swap contracts in fact are written as a contract on the quadratic variation of the $\log s_t$ process because i) this is a quantity that could be observed at the market, and ii) for models with no jumps the quadratic variance exactly coincides with the realized variance.

As shown by Hong [12], this fixed payment can be determined in any model where one has knowledge of the characteristic function of the future return $s_{t_i} - s_{t_{i-1}}$. The idea is as follows.

Let us define a forward characteristic function

$$\phi_{t,T} \equiv \mathbb{E}_{\mathbb{Q}}\left[\exp(ius_{t,T})|s_0, \nu_0\right] \equiv \int_{-\infty}^{\infty} e^{ius} q_{t,T}(s) ds, \tag{5}$$

where $s_{t,T} = s_T - s_t$ and $q_{t,T}$ is the $\mathbb{Q}$-density of $s_{t,T}$ conditional on the initial time state

$$q_{t,T}(s)ds \equiv \mathbb{Q}\left(s_{t,T} \in [s, s+ds)|s_0\right). \tag{6}$$

From Eq. (4) and Eq. (6) we obtain

$$\begin{aligned} \mathcal{Q}_N(s) &\equiv \frac{1}{T}\sum_{i=1}^{N}\mathbb{E}_{\mathbb{Q}}\left[(s_{t_i} - s_{t_{i-1}})^2\right] = \frac{1}{T}\sum_{i=1}^{N}\mathbb{E}_{\mathbb{Q}}\left[s^2_{t_i,t_{i-1}}\right] \\ &= -\frac{1}{T}\sum_{i=1}^{N}\frac{\partial^2\phi_{t_i,t_{i-1}}(u)}{\partial u^2}\Big|_{u=0}. \end{aligned} \tag{7}$$

Thus, if one knows the forward characteristic function of each discrete time increment of the price, one can use the above formula to compute the fixed payment on a variance swap per dollar of notional. The variance swap rate that one quotes is just the square root of this fixed payment.

3 Analytical expression for the forward characteristic function

According to Carr and Wu [11] consider a d-dimensional real-valued stochastic process $X_t|t \geq 0$ with $X_0 = 0$ defined on an underlying probability space $(Q, \mathfrak{F}, P)$ endowed with a standard complete filtration $\mathbf{F} = \{\mathfrak{F}_t|t \geq 0\}$. We assume that X is a Lévy process with respect to the filtration $\mathbf{F}$. That is, X_t is adapted to $\mathfrak{F}_t$, the sample paths of X are right-continuous with left limits, and $X_u - X_t$ is independent of $\mathfrak{F}_t$ and distributed as X_{u-t} for $0 \leq t \leq u$. The characteristic function of X_t then is given by the Lévy-Khintchine theorem (see [13]).

Next, let $t \to T_t(t \geq 0)$ be an increasing right-continuous process with left limits such that for each fixed t the random variable T_t is a stopping time with respect to $\mathbf{F}$. Suppose furthermore that T_t is finite P-a.s. for all $t \geq 0$ and that $T_t \to \infty$ as $t \to \infty$. Then the family of stopping times T_t defines a random time change. Without loss of generality, we further normalize the random time change so that $\mathbb{E}[T_t] = t$. With this normalization, the family of stopping times is an unbiased reflection of calendar time.

Finally, consider the d-dimensional process Y obtained by evaluating X at T, i.e., $Y_t \equiv X_{T_t}, t \geq 0$. We consider that this process describe the underlying uncertainty of the economy. For example, in the one-dimensional case, we can take Y as describing the returns on the asset underlying an option. Obviously, by specifying different Lévy characteristics for X_t and different random processes for T_t we can generate various stochastic processes from this setup. In principle, the random time T_t can be modeled as a non-decreasing semi-martingale.

In what follows we model dynamics of our underlying spot price by this kind of time-changed Lévy process, so that the log return follows the following equation

$$s_t \equiv \ln S_t/S_0 = (r - q)t + Y_t, \tag{8}$$

where r is the forward interest rate and q is the continuous dividend.

To remind, a general Lévy process X_t has its characteristic function represented in the form

$$\phi_X(u) = \mathbb{E}_{\mathbb{Q}}\left[e^{iuX_t}\right] = e^{-t\Psi_x(u)}, \tag{9}$$

where $\Psi_x(u)$ is known as a Lévy characteristic exponent ([13]).

For time-changed Lévy process, Carr and Wu (2004) show that the generalized Fourier transform can be converted into the Laplace transform of the time change under a new, complex-valued measure, i.e. the time-changed process $Y_t = X_{\mathbf{T}_t}$ has the characteristic function

$$\phi_{Y_t}(u) = \mathbb{E}_{\mathbb{Q}}\left[e^{iuX_{\mathbf{T}_t}}\right] = \mathbb{E}_{\mathbb{M}}\left[e^{-\mathbf{T}_t\Psi_x(u)}\right] = \mathcal{L}^u_{\mathbf{T}_t}\left(\Psi_x(u)\right), \tag{10}$$

where the expectation and the Laplace transform are computed under a new complex-valued measure $\mathbb{M}$. The measure $\mathbb{M}$ is absolutely continuous with respect to the risk-neutral measure $\mathbb{Q}$ and is defined by a complex-valued exponential martingale

$$\mathbb{D}_T(u) \equiv \frac{d\mathbb{M}}{d\mathbb{Q}}\Big|_T = \exp\left[iuY_T + \mathbf{T}_T\Psi_x(u)\right], \tag{11}$$

where $\mathbb{D}_T$ is the Radon-Nikodym derivative of the new measure with respect to the risk neutral measure up to time horizon T. Moreover, optimal stopping theorem ensures that

$$\mathbb{D}_t(u) = \mathbb{E}_{\mathbb{Q}}\left[\mathbb{D}_T(u)\middle|\mathcal{F}_t\right] = \exp\left[iuY_t + \mathbf{T}_t\Psi_x(u)\right] \tag{12}$$

is a $\mathbb{Q}$ martingale and that for all $\mathcal{F}_t$ random variable $\mathbf{Z}_T$ follows

$$\mathbb{E}_{\mathbb{M}}\left[\mathbf{Z}_T\middle|\mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{\mathbb{M}_T}{\mathbb{M}_t}\mathbf{Z}_T\middle|\mathcal{F}_t\right]. \tag{13}$$

Equation (11) reduces the problem of obtaining a generalized Fourier transform of a time-changed Lévy process into a simpler problem of deriving the Laplace transform of the stochastic clock. The solution to this Laplace transform depends on the specification of the instantaneous activity rate $\nu(t)$ and on the characteristic exponents.

Further we again follow the idea of Hong [12]. For the process Eq. (9) we need to obtain the forward characteristic function which is

$$\phi_{t,T}(u) \equiv \mathbb{E}_{\mathbb{Q}}\left[e^{iu(\log s_T - \log s_t)}\middle|\mathcal{F}_t\right] = e^{iu(r-q)(T-t)}\mathbb{E}_{\mathbb{Q}}\left[e^{iu(\mathbf{Y}_T-\mathbf{Y}_t)}\middle|\mathcal{F}_t\right], \tag{14}$$

where $t < T$. First, let us consider a single time-change process. The results for a vector version could be obtained in a similar way. From the Eq. (13) one has

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[e^{iu(\mathbf{Y}_T-\mathbf{Y}_t)}\middle|\mathcal{F}_t\right] &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[e^{iu(\mathbf{Y}_T-\mathbf{Y}_t)}\middle|\mathcal{F}_t\right]\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[e^{iu(\mathbf{Y}_T-\mathbf{Y}_t)+(\mathbf{T}_T-\mathbf{T}_t)\Psi_x(u)-(\mathbf{T}_T-\mathbf{T}_t)\Psi_x(u)}\middle|\mathcal{F}_t\right]\right] \\ &= \left[\mathbb{E}_{\mathbb{Q}}\left[\frac{\mathbb{M}_T}{\mathbb{M}_t}e^{-(\mathbf{T}_T-\mathbf{T}_t)\Psi_x(u)}\middle|\mathcal{F}_t\right]\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{M}}\left[e^{-(\mathbf{T}_T-\mathbf{T}_t)\Psi_x(u)}\middle|\mathcal{F}_t\right]\right]\end{aligned} \tag{15}$$

For Markovian arrival rates ν the inner expectation will be a function of $\nu(t)$ only.

Now let us consider a time-homogeneous time-change processes, for instance, CIR process with constant coefficients (as it is later specified in the Eq. (33)). With the allowance for the Eq. (10) the last expression could be rewritten as

$$\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{M}}\left[e^{-(\mathbf{T}_T-\mathbf{T}_t)\Psi_x(u)}\middle|\mathcal{F}_t\right]\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{M}}\left[e^{-\Psi_x(u)\int_t^T \nu(s)ds}\middle|\nu_t\right]\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathcal{L}_\theta^u\left(\Psi_x(u)\right)\middle|\nu_t\right],$$

where $\theta = \int_t^T \nu(s)ds$.

Now for all the arrival rates that are affine, the Laplace transform $\mathcal{L}_\theta^u\left(\Psi_x(u)\right)$ is also an exponential affine function in ν_t

$$\mathcal{L}_\theta^u\left(\Psi_x(u)\right) = \exp\left[\alpha(\tau,\Psi_x(u)) + \beta(\tau,\Psi_x(u))\nu_t\right], \quad \tau \equiv T - t. \tag{16}$$

and hence

$$\begin{aligned}e^{-iu(r-q)\tau}\phi_{t,T}(u) &= \mathbb{E}_{\mathbb{Q}}\left[e^{iu(\mathbf{Y}_T-\mathbf{Y}_t)}\middle|\mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}}\left[\exp\left[\alpha(\tau,\Psi_x(u)) + \beta(\tau,\Psi_x(u))\nu_t\right]\middle|\nu_t\right] \\ &= e^{\alpha(\tau,\Psi_x(u))}\mathbb{E}_{\mathbb{Q}}\left[e^{\beta(\tau,\Psi_x(u))\nu_t}\middle|\nu_t\right] = e^{\alpha(\tau,\Psi_x(u))}\phi_{\nu_t}\left(-i\beta(\tau,\Psi_x(u))\nu_t\right).\end{aligned} \tag{17}$$

Here as $\phi_{\nu_t}()$ we denote the generalized characteristic function of the activity rate process under the risk neutral measure $\mathbb{Q}$. If this characteristic function is available in a closed form as well as the characteristic exponent of the Lévy process, then one can use the Eq. (7) with $t = t_{i-1}$, $T = t_i$ (so $\theta = \int_{t_{i-1}}^{t_i}\nu(s)ds$) and get an analytical expression for the quadratic variation of the Lévy process $\mathcal{Q}_N(s)$. Actually if the arrival rate is affine then $\phi_{\nu_t}()$ is also an exponential affine function.

4 CIR clock change

Let consider two moments of time: t and $t+h, h>0$. In the case of the CIR clock change ($dy_t = \kappa(\theta - y_t)dt + \eta\sqrt{y_t}dZ_t$) the conditional Laplace transform (or moment generation function) of the CIR process

$$\psi_{t,h}(v) = \mathbb{E}_{\mathbb{Q}}\left[e^{-vy_{t+h}}\Big| y_t\right], \quad v \ge 0 \tag{18}$$

can be found in a closed form (see, for instance, [14]). Since ν_t in our case is a positive process, the conditional Laplace transform characterizes the transition between t and $t+h$ ([15]). The CIR is the affine process, therefore

$$\psi_{t,h}(v) = \mathbb{E}_{\mathbb{Q}}\left[e^{-vy_{t+h}}\Big| y_t\right] = \exp\left[-a(h,v)y_t - b(h,v)\right], \tag{19}$$

where functions a, b obey the differential equations

$$\begin{aligned} \frac{\partial a(h,v)}{\partial h} &= -\kappa a(h,v) - \frac{1}{2}\eta^2 a^2(h,v) \\ \frac{\partial b(h,v)}{\partial h} &= \kappa\theta a(h,v) \end{aligned} \tag{20}$$

with initial conditions $a(0,v) = v, b(0,v) = 0$.

This system of equations has the following solution

$$\begin{aligned} a(h,v) &= \frac{ve^{-\kappa h}}{1 + v\frac{\eta^2}{2\kappa}(1-e^{-\kappa h})} \\ b(h,v) &= \frac{2\kappa\theta}{\eta^2}\log\left[1 + v\frac{\eta^2}{2\kappa}(1-e^{-\kappa h})\right]. \end{aligned} \tag{21}$$

Now we apply these results to the Eq. (17). We substitute $\tau = t_i - t_{i-1}$ instead of h in the above expression and from the Eq. (19) obtain

$$\mathbb{E}_{\mathbb{Q}}\left[e^{\beta(\tau,\Psi_x(u))\nu_t}\Big|\nu_t\right] = \exp\Big[-a(\tau, -\beta(\tau,\Psi_x(u)))\nu_t - b(\tau, -\beta(\tau,\Psi_x(u)))\Big], \tag{22}$$

where coefficients a, b are given in the Eq. (21). Therefore, from the Eq. (17)

$$\phi_{t_{i-1},t_i}(u) = \exp\Big[\alpha(\tau,\Psi_x(u)) - a(\tau,-\beta(\tau,\Psi_x(u)))\nu_t - b(\tau,-\beta(\tau,\Psi_x(u)))\Big]. \tag{23}$$

Now, expressions for $\alpha(\tau,\Psi_x(u))$ and $\beta(\tau,\Psi_x(u))$ in the case of the CIR time-change have been already found in [11] and read

$$\begin{aligned} \beta(\tau,\Psi_x(u)) &= -\frac{2\Psi_x(u)(1-e^{-\delta\tau})}{(\delta+\kappa^Q) + (\delta-\kappa^Q)e^{-\delta\tau}}, \\ \alpha(\tau,\Psi_x(u)) &= -\frac{\kappa^Q\theta}{\eta^2}\left[2\log\left(1 - \frac{\delta-\kappa^Q}{2\delta}(1-e^{-\delta\tau})\right) + (\delta-\kappa^Q)\tau\right], \end{aligned} \tag{24}$$

where $\delta^2 = (\kappa^Q)^2 + 2\Psi_x(u)\eta^2$, $\kappa^Q = \kappa - iu\eta\sigma\rho$ and σ is a constant volatility rate of the diffusion component of the process[2].

[2] In [11] the authors do not discuss which branch of the complex logarithm function should be used in the above expression

Further let us have a more close look at the Eq. (7). Suppose the distance between any two observations at time t_{i-1} and t_i is one day. As it is well-known, taking into account an accurate business calendar brings just small corrections to the final quadratic variation value. Therefore, suppose also that these observations occur with no weekends and holidays. Then $\tau_i \equiv t_i - t_{i-1} = \tau = const$. Further we have to use the Eq. (23) with $t = t_{i-1}$ and $T = t_i$, substitute it into the Eq. (7), take second partial derivative and put $u = 0$. As this results in a very tedious algebra we use a simple Mathematica program while the resulting expression is still very bulky. To simplify it and make a qualitative analysis of the results transparent below we propose the following asymptotic method to obtain an approximate price of the quadratic variation swap contract.

4.1 Asymptotic method

A detailed analysis of the Eq. (23) shows that the time interval τ enters this equation only as a product $\kappa\tau$.

To prove it, note that according to the Eq. (7) we need to compute a second derivative of $\phi_{t_i,t_i+\tau}$ at $u = 0$. If we expand the Eq. (23) into series on u up to the quadratic term, the double coefficient at u^2 is just $\phi_{t_i,t_i+\tau}(u = 0)$. One can validate, for instance with Mathematica, that in the obtained expression the time interval τ appears only as a product $\kappa\tau$. Intuitively, this could be understood because in the Eq. (23) for $\phi_{t_i,t_i+\tau}$ the time interval τ appears only either as $\delta\tau$ or $\kappa\tau$. And according to the definition of δ given after Eq. (24) at $u = 0$ $\delta \equiv \kappa$ because $\Psi_x(0) = 0$. As the expression for $\phi_{t_i,t_i+\tau}$ has to contain $(\delta'_u)^2(u = 0)$ and $\delta''(u)(u = 0)$, one could expect to see some other terms proportional to τ, like $\tau\eta^2/\kappa$ etc.

Now we introduce an important observation that usually $\kappa\tau \ll 1$. Indeed, according to the results obtained for the Heston model calibrated to the market data the value of the mean-reversion coefficient κ lies in the range $0.01 - 30$. On the other hand, as it was already mentioned, we assume the distance between any two observations at time t_i and t_{i-1} to be one day, i.e $\tau = 1/365$. Therefore, the assumption $\kappa\tau \ll 1$ is provided with a high accuracy.

The above means that our problem of computing $\phi''_u(t_i, t_i + \tau)(u = 0)$ has two small parameters - u and $\kappa\tau$. And, in principal, we could produce a double series expansion of $\phi''_u(t_i, t_i + \tau)$ on both these parameters. However, to make it more transparent, let us expand the Eq. (23) first into series on $\kappa\tau$ up to the linear terms (that can also be done with Mathematica). Eventually we arrive at the following result

$$
-\frac{\partial^2 \phi_{t_i,t_{i-1}}(u)}{\partial u^2}\Big|_{u=0} = \xi\left[\theta + (\nu_0 - \theta)e^{-\kappa t_i}\right]\tau + O(\tau^2) \tag{25}
$$
$$
\xi = \frac{\partial^2 \Psi_x(u)}{\partial u^2}\Big|_{u=0}
$$

Then from the Eq. (7) we obtain

$$
\begin{aligned}
\mathcal{Q}_N(s) &= -\frac{1}{T}\sum_{i=1}^{N} \frac{\partial^2 \phi_{t_i,t_{i-1}}(u)}{\partial u^2}\Big|_{u=0} \approx \frac{\xi}{T}\int_0^T \left[\theta + (\nu_0 - \theta)e^{-\kappa t}\right] dt \\
&= \xi\left[\theta + (\nu_0 - \theta)\frac{1 - e^{-\kappa T}}{\kappa T}\right].
\end{aligned} \tag{26}
$$

4.2 Some examples

Heston model The above expression could be easily recognized if we remind that the familiar Heston model can be treated as the pure continuous Lévy component (pure lognormal diffusion process) with $\sigma = 1$ under the CIR time-changed clock. For the continuous diffusion process the characteristic exponent is (see, for instance, in [11]) $\Psi_x(u) = \sigma^2 u^2/2$, therefore for the Heston model $(\Psi_x)''_u(0) = 1$ [3] and

$$\mathcal{Q}_N(s) = \theta + (\nu_0 - \theta)\frac{1 - e^{-\kappa T}}{\kappa T} \tag{27}$$

Thus, we arrive at the well-known expression of the quadratic variation under the Heston model (see, for instance, [8]).

As it is seen from the Eq. (26) adding jump components to the description of the underlying stochastic process does not change the ansatz of the dependence $\mathcal{Q}_N(s,T)$ affecting only the coefficient of the ansatz. This looks to be a new and interesting result. Thus, the dependence $\mathcal{Q}_N(s,T)$ is basically determined by the stochastic time-change process, rather than by the Lévy model of the process.

To make it more transparent we can switch some steps in the derivation of the Eq. (27). Indeed, let us now first expand $\phi_{t,T}(u)$ in the Eq. (17) into series on $\kappa\tau$ and $(r-q)\tau$ (as the interest rate is usually about 1-10% and $\tau = 1/365$ that the first term is also a small parameter), that yields

$$\phi_{t,T}(u) = \mathbb{E}_{\mathbb{Q}}\left[1 - \tau\ \nu_t \Psi_x(u) + O(\tau^2)\middle|\nu_0\right]. \tag{28}$$

Then substituting this expression into the Eq. (7) we obtain

$$\mathcal{Q}_N(s) = \tau\xi \sum_{i=1}^{N} \mathbb{E}_{\mathbb{Q}}\left[\nu_{t_i}|\nu_0\right] \approx \xi \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \nu_t dt \mid \nu_0\right] \equiv \xi \mathbb{E}_{\mathbb{Q}}[V]. \tag{29}$$

The r.h.s. of this formula differs from the definition of the realized variance just by the constant coefficient ξ. Therefore, if one uses the CIR stochastic clock - same what is used in the Heston model - the resulting expression for $\mathcal{Q}_N(s)$ will differ from that for the Heston model by the same coefficient ξ. That is exactly what we obtained above.

SSM model According to Carr and Wu [11] let us consider a class of models that are known to be able to capture at least the average behavior of the realized volatilities of the stock price across moneyness and maturity.

We use $(\Omega, \mathcal{F}_t, \mathbb{Q})$ to denote a complete stochastic basis defined on a risk-neutral probability measure $\mathbb{Q}$ under which the log return obeys a time-changed Lévy process

$$s_t \equiv \log S_t/S_0 = (r-q)t + \left(\mathbf{L}^R_{T^R_t} - \xi^R \mathbf{T}^R_t\right) + \left(\mathbf{L}^L_{T^L_t} - \xi^L \mathbf{T}^L_t\right), \tag{30}$$

where r, q denote continuously-compounded interest rate and dividend yield, both of which are assumed to be deterministic; $\mathbf{L}^R$ and $\mathbf{L}^L$ denote two Lévy processes that exhibit right (positive) and left (negative) skewness respectively; T^R_t and T^L_t denote two separate stochastic time changes applied to the Lévy components; ξ^R and ξ^L are known functions of the parameters governing these Lévy processes, chosen so that the exponentials of $\mathbf{L}^R_{T^R_t} - \xi^R \mathbf{T}^R_t$ and $\mathbf{L}^L_{T^L_t} - \xi^L \mathbf{T}^L_t$ are both $\mathbb{Q}$ martingales. Each Lévy component can has a diffusion component, and both must have a jump component to generate the required skewness.

Carr and Wu notice that in principle, this generic specification can capture all of the documented features, for instance, of the currency options. First, by setting the unconditional weight of the two Lévy components equal to each other, we can obtain an unconditionally symmetric distribution with fat tails for the currency return under the risk-neutral measure. This unconditional property captures the relative symmetric feature of the sample averages of the implied volatility smile. Second,

[3] The second derivative of the drift term $-i(r-q)u$ on u vanishes.

by applying separate time changes to the two components, aggregate return volatility can vary over time so that the model can generate stochastic volatility. Third, the relative weight of the two Lévy components can also vary over time due to the separate time change. When the weight of the right-skewed Lévy component is higher than the weight of the left-skewed Lévy component, the model generates a right-skewed conditional return distribution and hence positive risk reversals. When the opposite is the case, the model generates left-skewed conditional return distribution and negative risk reversals. Thus, we can generate variations and even sign changes on the risk reversals via the separate time change. Finally, the model captures the instantaneous correlation between the return and the risk reversal through the correlations between the Lévy components and the time change.

For model design we make the following decomposition of the two Lévy components in the Eq. (30)

$$\mathbf{L}_t^R = J_t^R + \sigma^R W_t^R, \quad \mathbf{L}_t^L = J_t^L + \sigma^L W_t^L, \tag{31}$$

where (W_t^R, W_t^L) denote two independent standard Brownian motions and (J_t^R, J_t^L) denote two pure Lévy jump components with right and left skewness in distribution, respectively.

We assume a differentiable and therefore continuous time change and let

$$\nu_t^R \equiv \frac{\partial \mathbf{T}_t^R}{\partial t}, \quad \nu_t^L \equiv \frac{\partial \mathbf{T}_t^L}{\partial t}, \tag{32}$$

denote the instantaneous activity rates of the two Lévy components. By definition $\mathbf{T}_t^R, \mathbf{T}_t^L$ have to be non-decreasing semi-martingales. We model the two activity rates as a certain affine process. For instance, it could be a square-root processes of Heston [16]

$$\begin{aligned} d\nu_t^R &= \kappa^R(\theta^R - \nu_t^R)dt + \eta^R \sqrt{\nu_t^R} dZ_t^R, \\ d\nu_t^L &= \kappa^L(\theta^L - \nu_t^L)dt + \eta^L \sqrt{\nu_t^L} dZ_t^L, \end{aligned} \tag{33}$$

where in contrast to [11] we don't assume unconditional symmetry and therefore use different mean-reversion κ, long-run mean θ and volatility of volatility η parameters for left and right activity rates.

We allow the two Brownian motions (W_t^R, W_t^L) in the return process and the two Brownian motions (Z_t^R, Z_t^L) in the activity rates to be correlated as follows,

$$\rho^R dt = \mathbb{E}_{\mathbb{Q}}[dW_t^R dZ_t^R], \quad \rho^L dt = \mathbb{E}_{\mathbb{Q}}[dW_t^L dZ_t^L]. \tag{34}$$

The four Brownian motions are assumed to be independent otherwise.

Note that the above definition is pretty wide in sense that it covers a lot of the existing models, including Merton jump-diffusion model, Kou double-exponential model, NIG, VG, CGMY, Hyperbolic, LS and even pure continuous models.

Now assuming that the positive and negative jump components are driven by two different CIR stochastic clocks as in the Eq. (33), it could be shown in exactly same way as we did for the single time process, that the annualized fair strike $\mathcal{Q}_N(s,T)$ is now given by the expression

$$\mathcal{Q}_N(s,T) = \xi^L \left[\theta^L + (\nu_0^L - \theta^L) \frac{1 - e^{-\kappa^L T}}{\kappa^L T} \right] + \xi^R \left[\theta^R + (\nu_0^R - \theta^R) \frac{1 - e^{-\kappa^R T}}{\kappa^R T} \right]. \tag{35}$$

So now we have two independent mean-reversion rates and two long-term run coefficients that can be used to provide a better fit for the long-term volatility level and the short-term volatility skew, similar to how this is done in the multifactor Heston (CIR) model.

4.3 Volatility swaps

Similar to a contract on the quadratic variation, a volatility swap contract makes a bet on the annualized realized volatility that is defined as follows

$$\overline{Vol}(s_t) \equiv \frac{1}{T}\mathbb{E}_{\mathbb{Q}}\sqrt{\sum_{i=1}^{N}\left[\ln\frac{s_{t_i}}{s_{t_{i-1}}}\right]^2} \approx \frac{1}{T}\mathbb{E}_{\mathbb{Q}}\left[\sqrt{\int_0^T \nu_t dt} \;\Big|\; \nu_0\right] = \frac{1}{T}\mathbb{E}_{\mathbb{Q}}[\sqrt{V}], \tag{36}$$

where V stays for the total annualized realized variance.

Swishchuk [8] uses the second order Taylor expansion for function $\sqrt{V}$ obtained in [17] to represent $\mathbb{E}_{\mathbb{Q}}[\sqrt{V}]$ via $\mathbb{E}_{\mathbb{Q}}[V]$ and $Var[V]$ as

$$\mathbb{E}_{\mathbb{Q}}[\sqrt{V}] \approx \sqrt{\mathbb{E}_{\mathbb{Q}}[V]} - \frac{VarV}{8(\mathbb{E}_{\mathbb{Q}}[V])^{3/2}}. \tag{37}$$

As we already showed in the Eq. (29) for the CIR time-change the quadratic variation process V differs from that of the Heston model by the constant coefficient ξ. Therefore, $Var[V]$ in our case differs from that for the Heston model by the coefficient ξ^2. As Swishchuk and Brokchaus and Long showed for the Heston model

$$Var[V] = \frac{\eta^2 e^{-\kappa T}}{2\kappa^3 T^2}\left[\left(2e^{2\kappa T} - 4\kappa T e^{\kappa T} - 2\right)(\nu_0 - \theta) + \theta\left(2\kappa T e^{2\kappa T} - 3e^{2\kappa T} + 4e^{\kappa T} - 1\right)\right]. \tag{38}$$

Thus, for the Lévy models with the CIR time-change the fair value of the annualized realized volatility is

$$\overline{Vol}(s_t) = \sqrt{\xi}\;\overline{Vol}_H(s_t), \tag{39}$$

where $\overline{Vol}_H(s_t)$ is this value for the Heston model obtained by using the Eq. (37), 38 and 27.

A more rigorous approach is given by Jim Gatheral [18]. He uses the following exact representation

$$\mathbb{E}_{\mathbb{Q}}\left[\sqrt{V}\right] = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{1 - \mathbb{E}_{\mathbb{Q}}\left[e^{-xV}\right]}{x^{3/2}}dx. \tag{40}$$

Here

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-xV}\right] = \mathbb{E}_{\mathbb{Q}}\left[exp\left\{-x\int_0^T v_t dt\right\}\right]$$

is formally identical to the expression for the value of a bond in the CIR model (Eq. (19)) if one substitutes there $\beta(\tau, \Psi_x(u))$ with $-x$.

4.4 Options on the quadratic variation

Having known the values of $\mathbb{E}_{\mathbb{Q}}[V]$ and $\mathbb{E}_{\mathbb{Q}}[\sqrt{V}]$ we can price vanilla European options on the quadratic variation using a log-normal method of Jim Gatheral [5]. This method, however, first is an approximation, and second, for complicated models like SSM, accurate computing of $\mathbb{E}_{\mathbb{Q}}[\sqrt{V}]$ could be a problem. Also we found after a careful consideration that another method proposed in [4] results just in a true identity, and thus cannot be used for obtaining the option value. Therefore we intend to proceed in sense of Roger Lee paper [19] and make use of the FFT method.

Let us denote

$$Q(T) \equiv \lambda\int_0^T \nu_t dt, \quad \lambda \equiv (\Psi_x)''_u(0)\frac{1}{T}. \tag{41}$$

For the CIR process the characteristic function $\phi(u,T) \equiv \mathbb{E}_{\mathbb{Q}}[e^{iuQ(T)}]$ is known

$$\begin{aligned} \phi(u,T) &= Ae^B, \\ B &= \frac{2iu\lambda v_0}{\kappa + \delta \coth(\delta T/2)}, \\ A &= \exp\left[\frac{\kappa^2 \theta T}{\eta^2}\right] \left[\cosh(\delta T/2) + \frac{\kappa}{\delta} \sinh(\delta T/2)\right]^{-\frac{2\kappa\theta}{\eta^2}} \\ \delta^2 &= \kappa^2 - 2iu\lambda\eta^2. \end{aligned} \tag{42}$$

Therefore, according to [19] the call option value on the quadratic variation is given by the following integral

$$C(K,T) = \frac{e^{-\alpha \log(K)}}{\pi} \int_0^\infty \mathrm{Re}\left[e^{-iv\log(K)}\omega(v)\right] dv, \tag{43}$$

where

$$\omega(v) = \frac{e^{-rT}\phi(v - i\alpha, T)}{(\alpha + iu)^2} \tag{44}$$

The integral in the first equation can be computed using FFT, and as a result we get call option prices for a variety of strikes. For complete details see [19] and Carr and Madan original paper [9].

The put option values can just be constructed from the Put/Call symmetry.

Parameter α in the Eq. (43) must be positive. Usually $\alpha = 0.75$ works well for various models. It is important that the denominator in Eq. (44) has only imaginary roots while integration in Eq. (43) is provided along real v. Thus, the integrand of Eq. (43) is well-behaved.

Note that a similar approach was proposed in [20].

5 Other affine activity rates models

Further we follow Carr and Wu [11] to consider affine activity rate models with more general jump specification. First they prove the following proposition.

Proposition 5.1 (Carr-Wu). *If the instantaneous activity rate ν_t, the drift vector $\mu(Z)$, the diffusion covariance matrix $\sigma(Z)\sigma^\top(Z)$, and the arrival rate $\gamma(Z)$ of the Markov process are all affine in Z then the Laplace transform $\mathcal{L}_{T_t}(\lambda)$ is exponential-affine in z_0.*

The assumptions of this proposition mean that the following representation takes place

$$\begin{aligned} \nu(t) &= \mathbf{b}_v^\top Z_t + c_v, \quad \mathbf{b} \in \mathbb{R}^k, c_v \in \mathbb{R} \\ \mu(Z_t) &= a - kZ_t, \quad k \in \mathbb{R}^{kxk}, a \in \mathbb{R}^k \\ \left[\sigma(Z_t)\sigma(Z_t)^\top\right]_{ii} &= \alpha_i + \beta_i^\top Z_t, \quad \alpha_i \in \mathbb{R}, \beta_i \in \mathbb{R}^k \\ \left[\sigma(Z_t)\sigma(Z_t)^\top\right]_{ij} &= 0, \quad i \neq j \\ \gamma(Z_t) &= a_\gamma + \mathbf{b}_\gamma^\top Z_t, \quad \alpha_\gamma \in \mathbb{R}, \beta_\gamma \in \mathbb{R}^k \end{aligned} \tag{45}$$

In a one factor setting Carr and Wu adopt a generalized version of the affine term structure model proposed by Filipovic [21], which allows a more flexible jump specification. The activity rate process

ν_t is a Feller process with generator

$$\mathcal{A}f(x) = \frac{1}{2}\sigma^2 x f''(x) + (a' - kx)f'(x) \tag{46}$$
$$+ \int_{\mathbb{R}_0^+} \left[f(x+y) - f(x) - f'(x)(1 \wedge y)\right] (m(dy) + x\mu(dy)),$$

where $a' = a + \int_{\mathbb{R}_0^+}(1 \wedge y)m(dy)$ for some constant numbers $\sigma, a \in \mathbb{R}^+, k \in \mathbb{R}^+$ and nonnegative Borel measures $m(dy)$ and $\mu(dy)$ satisfying the following condition:

$$\int_{\mathbb{R}_0^+}(1 \wedge y)m(dy) + \int_{\mathbb{R}_0^+}(1 \wedge y^2)\mu(dy) < \infty. \tag{47}$$

The first line in Eq. (46) is due to the continuous part of the process and is equivalent to the Cox et al. [22] or Heston [16] specification. The second line is due to the jump part of the process. All three components of the Lévy triplet depend linearly on the state variable x. The condition in Eq. (47) says that the jump component dictated by the measure $m(dy)$ has to exhibit finite variation, while the jump component dictated by the measure $\mu(dy)$ only needs to exhibit finite quadratic variation. Carr and Wu provide various Lévy measure specifications that can be adopted with the only slight modification: arrival rates of negative jumps need to be set to zero to have the stochastic clock to be a Lévy subordinator.

Under such a specification, the Laplace transform of random time is exponential

$$\mathcal{L}^u_{\mathbf{T}_t}(\Psi_x(u)) = \exp\left[-\alpha(t, \Psi_x(u)) - \beta(t, \Psi_x(u))\nu_t\right], \tag{48}$$

with the coefficients $\alpha(t, \Psi_x(u)), \beta(t, \Psi_x(u))$ given by the following ordinary differential equations:

$$\beta'_t(t, \Psi_x(u)) = \Psi_x(u) - k\beta(t, \Psi_x(u)) - \frac{1}{2}\sigma^2\beta^2(t, \Psi_x(u)) \tag{49}$$
$$+ \int_{\mathbb{R}_0^+}\left[1 - e^{-y\beta(t,\Psi_x(u))} - \beta(t, \Psi_x(u))(1 \wedge y)\right]\mu(dy),$$
$$\alpha'_t(t, \Psi_x(u)) = a\beta(t, \Psi_x(u)) + \int_{\mathbb{R}_0^+}\left[1 - e^{-y\beta(t,\Psi_x(u))}\right] m(dy),$$

with boundary conditions $\beta(0) = \alpha(0) = 0$.

Now we are ready to formulate a more general result for the quadratic variation of such Lévy processes.

Theorem 1. *Given the above conditions the annualized quadratic variation of the Lévy process under stochastic time determined by a pure diffusion process is*

$$\mathcal{Q}_N(s) = \frac{1}{T}\xi\mathbb{E}_{\mathbb{Q}}[V], \tag{50}$$
$$\xi \equiv (\Psi_x)''_u(0)\frac{\partial^2\beta(t, \Psi_x(u))}{\partial t \partial u}\Big|_{t,u=0} + (\Psi_x)'^2_u(0)\frac{\partial^3\beta(t, \Psi_x(u))}{\partial t \partial^2 u}\Big|_{t,u=0}.$$

Proof 1.

We prove it based on the idea considered in the previous sections. Namely, we again express $\mathcal{Q}_N(s)$ as in the Eq. (7) via the forward characteristic function $\phi_{t_{i-1},t_i}(u)$, which is (the drift term already appears as k in the Eq. (49))

$$\phi_{t_{i-1},t_i}(u) = \mathbb{E}_{\mathbb{Q}}\left[\mathcal{L}^u_{\mathbf{T}_\tau}(\Psi_x(u)) \;\middle|\; \nu_0\right] = \mathbb{E}_{\mathbb{Q}}\left[\exp\left[-\alpha(\tau, \Psi_x(u)) - \beta(\tau, \Psi_x(u))\nu_t\right] \;\middle|\; \nu_0\right]. \tag{51}$$

Let us remind the reader that $\kappa\tau \ll 1$ is a small parameter as well as $(r-q)\tau \ll 1$. Therefore we expand the above expression in series on τ up to the linear terms to obtain

$$\phi_{t_{i-1},t_i}(u) = \mathbb{E}_{\mathbb{Q}}\Big\{ \exp\left[-\alpha(0,\Psi_x(u)) - \beta(0,\Psi_x(u))\nu_t\right] \cdot \tag{52}$$
$$\left[1 - \left(\frac{\partial\alpha(\tau,\Psi_x(u))}{\partial\tau}\Big|_{\tau=0} + \frac{\partial\beta(\tau,\Psi_x(u))}{\partial\tau}\Big|_{\tau=0}\nu_t \right)\tau\right] + O(\tau^2).\Big\}$$

Note, that according to the boundary conditions $\alpha(0,\Psi_x(u)) = \beta(0,\Psi_x(u)) = 0$. Therefore, differentiating the above formula twice on u gives

$$\frac{\partial^2\phi_{t_{i-1},t_i}(u)}{\partial u^2}\Big|_{u=0} = -\tau\Big\{ \Psi_x''(0)\left[\frac{\partial^2\alpha(t,\Psi_x(u))}{\partial t\partial u}\Big|_{t,u=0} + \nu_t\frac{\partial^2\beta(t,\Psi_x(u))}{\partial t\partial u}\Big|_{t,u=0}\right] \tag{53}$$
$$+ (\Psi_x'(0))^2\left[\frac{\partial^3\alpha(t,\Psi_x(u))}{\partial t\partial^2 u}\Big|_{t,u=0} + \nu_t\frac{\partial^3\beta(t,\Psi_x(u))}{\partial t\partial^2 u}\Big|_{t,u=0}\right]\Big\}$$

Further it could be easily checked from the Eq. (49) and the boundary conditions that $\beta_t'(0,\Psi_x(0)) = 0$, $\alpha_t'(0,\Psi_x(0)) = 0$. Differentiating the second equation in the Eq. (49) we obtain that

$$\frac{\partial^2\alpha(t,\Psi_x(u))}{\partial t\partial u}\Big|_{t,u=0} = \frac{\partial\alpha}{\partial u}(0,\Psi_x(0))\beta(0,\Psi_x(0)) + \frac{\partial\beta}{\partial u}(0,\Psi_x(0))\alpha(0,\Psi_x(0)) \tag{54}$$
$$+ \int_{\mathbb{R}_0^+} y\frac{\partial\beta}{\partial u}(0,\Psi_x(0))e^{-y\beta(0,\Psi_x(0))}m(dy) = \frac{\partial\beta}{\partial u}(0,\Psi_x(0))\int_{\mathbb{R}_0^+} y\ m(dy)$$
$$\frac{\partial^3\alpha(t,\Psi_x(u))}{\partial t\partial^2 u}\Big|_{t,u=0} = \frac{\partial^2\alpha}{\partial u^2}(0,\Psi_x(0))\beta(0,\Psi_x(0)) + \frac{\partial^2\beta}{\partial u^2}(0,\Psi_x(0))\alpha(0,\Psi_x(0))$$
$$+ 2\frac{\partial\alpha}{\partial u}(0,\Psi_x(0))\frac{\partial\beta}{\partial u}(0,\Psi_x(0)) + \int_{\mathbb{R}_0^+} m(dy)e^{-y\beta(0,\Psi_x(0))}\Big[y\frac{\partial^2\beta}{\partial u^2}(0,\Psi_x(0))$$
$$- y^2\left(\frac{\partial\beta}{\partial u}(0,\Psi_x(0))\right)^2\Big] = \frac{\partial^2\beta}{\partial u^2}(0,\Psi_x(0))\int_{\mathbb{R}_0^+} y\ m(dy) - \left(\frac{\partial\beta}{\partial u}(0,\Psi_x(0))\right)^2\int_{\mathbb{R}_0^+} y^2\ m(dy)$$

Therefore, if the activity rate is a pure diffusion process these two derivatives of $\alpha(t,\Psi_x(u))$ vanish. Thus, we finally found that

$$\frac{\partial^2\phi_{t_{i-1},t_i}(u)}{\partial u^2}\Big|_{u=0} = -\mathbb{E}_{\mathbb{Q}}[\xi\tau\nu_t]. \tag{55}$$

Using this formula together with the Eq. (7) we obtain the result of the Theorem.

$$\mathcal{Q}_N(s) = -\frac{1}{T}\sum_{i=1}^{N}\frac{\partial^2\phi_{t_{i-1},t_i}(u)}{\partial u^2}\Big|_{u=0} \approx \frac{1}{T}\xi\mathbb{E}_{\mathbb{Q}}[\int_0^T \nu_t dt \mid \nu_0] = \frac{1}{T}\xi\mathbb{E}_{\mathbb{Q}}[V]. \tag{56}$$

■

Theorem 2. *Given the above conditions the annualized quadratic variation of the Lévy process under* *stochastic time determined by a jump diffusion process is*

$$
\begin{aligned}
\mathcal{Q}_N(s) &= \frac{1}{T}\xi \mathbb{E}_{\mathbb{Q}}[V] + \eta, \qquad (57)\\
\eta &\equiv (\Psi_x)''_u(0)\frac{\partial^2\alpha(t,\Psi_x(u))}{\partial t\partial u}\Big|_{t,u=0} + (\Psi_x)'^2_u(0)\frac{\partial^3\alpha(t,\Psi_x(u))}{\partial t\partial^2 u}\Big|_{t,u=0}\\
&= (\Psi_x)''_u(0)\frac{\partial\beta}{\partial u}(0,\Psi_x(0))\mathcal{I}_1 + (\Psi_x)'^2_u(0)\left[\frac{\partial^2\beta}{\partial u^2}(0,\Psi_x(0))\mathcal{I}_1 - \left(\frac{\partial\beta}{\partial u}(0,\Psi_x(0))\right)^2\mathcal{I}_2\right],\\
\mathcal{I}_n &\equiv \int_{\mathbb{R}_0^+} y^n\ m(dy).
\end{aligned}
$$

Proof 2. *The proof directly follows from the previous Theorem.*

Thus, in the case of jumps in the activity rate even the exponential affinity of the Laplace transform of the random time is not sufficient to provide the same result, i.e. the quadratic variation differs not just by a constant multiplier ξ from the total realized variance of the process but also by some constant η.

Now let us consider the case when discrete observations of the underlying spot price occur over a bigger time interval, such that $\kappa\tau$ might not be a small parameter.

Theorem 3. *Given the above conditions the annualized quadratic variation of the Lévy process under stochastic time determined by a pure diffusion process is*

$$
\mathcal{Q}_N(s) = \frac{1}{\tau}\left\{p_0(\tau) + \frac{p_1(\tau)}{T}\mathbb{E}_{\mathbb{Q}}[V] + \frac{p_2(\tau)}{T}\mathbb{E}_{\mathbb{Q}}[V^2]\right\} \qquad (58)
$$

Proof 3.

We start with the Eq. (51) and differentiating it twice on u obtain

$$
\begin{aligned}
\phi_{t_{i-1},t_i}(u)\Big|_{u=0} = \mathbb{E}_{\mathbb{Q}}\Bigg\{ &\exp\left[-\alpha(\tau,\Psi_x(0)) - \beta(\tau,\Psi_x(0))\nu_t\right]\cdot \qquad (59)\\
&\left[-\frac{\partial^2\alpha(\tau,\Psi_x(u))}{\partial u^2}\Big|_{u=0} - \frac{\partial^2\beta(\tau,\Psi_x(u))}{\partial u^2}\Big|_{u=0}\nu_t + \left(\frac{\partial\alpha(\tau,\Psi_x(u))}{\partial u}\Big|_{u=0} + \frac{\partial\beta(\tau,\Psi_x(u))}{\partial u}\Big|_{u=0}\nu_t\right)^2\right]\Bigg\}
\end{aligned}
$$

Show that $\alpha(\tau,\Psi_x(0)) = \beta(\tau,\Psi_x(0)) = 0$. Expanding $\alpha(\tau,\Psi_x(0))$ and $\beta(\tau,\Psi_x(0))$ in series on τ and noticing that $\Psi_x(0) = 0$ we have

$$
\begin{aligned}
\alpha(\tau,\Psi_x(0)) &= \alpha(0,0) + \tau\frac{\partial\alpha(\tau,0)}{\partial\tau} + \frac{1}{2}\tau^2\frac{\partial^2\alpha(\tau,0)}{\partial\tau^2} + \ldots \qquad (60)\\
\beta(\tau,\Psi_x(0)) &= \beta(0,0) + \tau\frac{\partial\alpha(\tau,0)}{\partial\tau} + \frac{1}{2}\tau^2\frac{\partial^2\beta(\tau,0)}{\partial\tau^2} + \ldots
\end{aligned}
$$

Now

1. $\alpha(0,0) = \beta(0,0) = 0$ according to the boundary conditions to the Eq. (49).
2. From the second equation of Eq. (49) it follows that in case of no jumps $\alpha'_\tau(\tau,0) = a\beta(\tau,0) = 0$. In turn from the first equation $\beta'_\tau(\tau,0) = 0$.
3. Differentiating the Eq. (49) in time and using a chain rule we arrive at the conclusion that all higher derivatives of $\alpha(\tau,0)$ and $\beta(\tau,0)$ in time vanish as well.

Thus the Eq. (59) could be rewritten as

$$-\phi_{t_{i-1},t_i}(u)\Big|_{u=0} = \mathbb{E}_{\mathbb{Q}}\left[p_0 + p_1\nu_t + p_2\nu_t^2\right], \tag{61}$$

$$p_0(\tau) = \frac{\partial^2\alpha(\tau,\Psi_x(u))}{\partial u^2}\Big|_{u=0} - \left[\frac{\partial\alpha(\tau,\Psi_x(u))}{\partial u}\Big|_{u=0}\right]^2$$

$$p_1(\tau) = \frac{\partial^2\beta(\tau,\Psi_x(u))}{\partial u^2}\Big|_{u=0} - 2\frac{\partial\alpha(\tau,\Psi_x(u))}{\partial u}\Big|_{u=0}\frac{\partial\beta(\tau,\Psi_x(u))}{\partial u}\Big|_{u=0}$$

$$p_2(\tau) = -\left[\frac{\partial\beta(\tau,\Psi_x(u))}{\partial u}\Big|_{u=0}\right]^2$$

And therefore

$$\mathcal{Q}_N(s) = \frac{1}{T}\sum_{i=1}^{N}\left\{p_0(\tau) + p_1(\tau)\mathbb{E}_{\mathbb{Q}}[\nu_{t_i}] + p_2(\tau)\mathbb{E}_{\mathbb{Q}}[\nu_{t_i}^2]\right\} \approx \frac{1}{T}\left\{Np_0(\tau) + \frac{p_1(\tau)}{\tau}\mathbb{E}_{\mathbb{Q}}[V] + \frac{p_2(\tau)}{\tau}\mathbb{E}_{\mathbb{Q}}[V^2]\right\}$$

$$= \frac{1}{\tau}\left\{p_0(\tau) + \frac{p_1(\tau)}{T}\mathbb{E}_{\mathbb{Q}}[V] + \frac{p_2(\tau)}{T}\mathbb{E}_{\mathbb{Q}}[V^2]\right\} \tag{62}$$

■

Thus if the time distance between market observations is not small the formula for the price of quadratic variation swap acquires two extra terms. The first one $p_0(\tau)$ is a function of time between observations τ and is determined by a particular model of the underlying Lévy process. The last term $p_2(\tau)\mathbb{E}_{\mathbb{Q}}[V^2]$ is proportional to the square of variance and is kind of convexity adjustment.

Based on this representation we could reconsider our results obtained in the previous sections, for instance for the CIR time change model. Expanding coefficients $p_0(\tau), p_1(\tau), p_2(\tau)$ into series on τ and keeping the first two terms gives

$$p_0(\tau) \approx -\frac{1}{2}\kappa\theta\frac{\partial^2\Psi_x(u)}{\partial u^2}\Big|_{u=0}\tau^2 + O(\tau^3) \tag{63}$$

$$p_1(\tau) \approx \frac{\partial^2\Psi_x(u)}{\partial u^2}\Big|_{u=0}\tau + \left[\frac{1}{2}\kappa\frac{\partial^2\Psi_x(u)}{\partial u^2}\Big|_{u=0} + \frac{2\eta^2}{\kappa}\left(\frac{\partial\Psi_x(u)}{\partial u}\Big|_{u=0}\right)^2\right]\tau^2 + O(\tau^3)$$

$$p_2(\tau) \approx -\left(\frac{\partial\Psi_x(u)}{\partial u}\Big|_{u=0}\right)^2\tau^2 + O(\tau^3)$$

This means that in the first approximation on $\kappa\tau$ coefficients $p_0(\tau)$ and $p_2(\tau)$ vanish. That is why the price of the quadratic variation swap is proportional to $\mathbb{E}_{\mathbb{Q}}[V]$, i.e. the standard log contract. However, $p_0(\tau)$ and $p_2(\tau)$ appear in the second order approximation of the price in $\kappa\tau$. For instance, for the Heston model (a pure diffusion underlying process) these coefficients read

$$p_0(\tau) = -\frac{1}{2}\kappa\theta\tau^2 + O(\tau^3), \qquad p_1(\tau) = \tau + \frac{1}{2}\kappa\tau^2 + O(\tau^3) \qquad p_2(\tau) = O(\tau^3). \tag{64}$$

In general it can be shown that for the CIR time change model and a pure diffusion underlying process the coefficient $p_2(\tau) = 0$.

6 3/2 power clock change

In this section we consider one more class of the stochastic clock change. Despite it is not affine, it still allows variance swaps to be priced in a closed form.

Originally this model has been proposed in a simple form (long term run coefficient is constant) by Heston [23] and Lewis [24] to investigate stochastic volatility. Here we consider a more general case when the long-term run could be either a deterministic function of time, or even a stochastic process.

Let the futures price F of the underlying asset be a positive continuous process. By the martingale representation theorem, there exists a process v such that:

$$\frac{dF_t}{F_t} = \sqrt{v_t} d\tilde{Z}_t, \qquad t \in [0, T], \tag{65}$$

where $\tilde{Z}$ is a $\mathbb{Q}$ standard Brownian motion. In particular, let us assume the risk-neutral process for instantaneous variance to be:

$$dv_t = \kappa v_t(\theta_t - v_t)dt + \epsilon v_t^{3/2} d\tilde{W}_t, \qquad t \in [0, T], \tag{66}$$

where $\tilde{W}$ is a $\mathbb{Q}$ standard Brownian motion, whose increments have known constant correlation $\rho \in [-1, 1]$ with increments in the $\mathbb{Q}$ standard Brownian motion $\tilde{Z}_t$, i.e.:

$$d\tilde{Z}_t d\tilde{W}_t = \rho dt, \qquad t \in [0, T]. \tag{67}$$

The risk-neutral process Eq. (66) for v has its volatility governed by the known positive constant ϵ. The 3/2 power specification for the volatility of v is empirically supported. The v process is mean-reverting with speed of mean reversion κv_t, where κ is known. The reason that the speed of mean-reversion is proportional rather than constant is primarily for tractability. In fact, when θ_t is a deterministic function of time (let us remind that Heston and Lewis explored just the case $\theta = const$), the process Eq. (66) is *more* tractable than the usual Heston dynamics, since we will show that there exists a closed form solution for the characteristic function of the log price. In contrast, for the Heston model where the long run mean θ_t is a deterministic function of time, there is no closed form formula for the characteristic function of the log price. As a bonus, when $v_0 > 0$ and the process θ is positive, then the process Eq. (66) neither explodes nor hits zero. In contrast, the Heston process can hit zero for some parameter values,which is unrealistic. Although our primary motivation for proportional speed of mean-reversion is tractability, nonlinear drift in the v process is also empirically supported.

In the Eq. (66), the level towards which v reverts is assumed to be an *unknown* stochastic process θ. We do assume however that θ is conditionally independent of the two $\mathbb{Q}$ standard Brownian motions $\tilde{Z}$ and $\tilde{W}$, i.e.:

$$d\theta_t d\tilde{Z}_t = 0 = d\theta_t d\tilde{W}_t, \qquad t \in [0, T]. \tag{68}$$

We also assume that the evolution coefficients of θ are independent of F and v and of the Brownian motions $\tilde{Z}_t$ and $\tilde{W}_t$ driving them. We may summarize these two assumptions by saying that θ evolves independently of F and v. Other than this independence assumption, we assume nothing about the dynamics of θ, not even its initial level. So long as consistency with independence is maintained, the process θ can jump, be arbitrarily path dependent, and can depend on other processes.

As a result, we call our pricing theory robust since the risk-neutral dynamics of θ are not fully specified. In the Black Scholes model, both θ and v are constant and equal to each other. The usual approach for introducing stochastic volatility is to specify a particular stochastic process for v and keep θ constant. Indeed, assuming that the risk-neutral process for instantaneous variance is:

$$dv_t = \kappa v_t(\theta - v_t)dt + \epsilon v_t^{3/2} d\tilde{W}_t, \qquad t \in [0, T], \tag{69}$$

with θ a known constant, Heston[23] and Lewis[24] solve for the characteristic function of X_T in closed form. Once the characteristic function of X_T is known in closed form, it is straightforward to numerically determine option prices.

In our model, one can interpret θ as the process that the instantaneous variance would follow if the speed of mean reversion were infinite. When κ is finite, the process θ is instead a state variable that governs the level of v. Without formally realizing it, Carr and Lee [3] consider the case where κ is infinite. When $\kappa = \infty$, the 3/2 specification for the volatility of variance and the correlation ρ between the two Brownian motions becomes irrelevant. Assuming an independent but otherwise unspecified stochastic process for v, Carr and Lee show how to replicate the payoff to volatility derivatives by dynamic trading in standard European options and their underlying futures. An unfortunate implication of their independence assumption is that implied volatility is always a symmetric function of the difference between log strike and log forward. In other words, their analysis is consistent with the existence of an implied volatility smile, but not an implied volatility skew. Since implied volatility tends to display both smile and skew in most markets, they add independent jumps of known size in order to generate a skew.

To be consistent with both smile and skew without introducing jumps, we employ a finite κ and perimetrically specify how v tends towards the unspecified process θ. As in Carr and Lee, our pricing of volatility derivatives maturing at T will be perfectly consistent with the implied volatility smile of maturity T. In contrast to Carr and Lee, our analysis requires that the parameters κ, ϵ, and ρ be known constants. Our analysis also requires the state variable v_0 be known in order to initially price volatility derivatives. For now, we just naively assume that κ, ϵ, ρ, and v_0 are somehow known. Future research will focus on the roles of κ, ϵ, ρ, and v_0 in calibrating across maturities or determining appropriate hedge ratios. It will also explore how the parameters might be learned from the time series or how these required inputs might be learned from knowledge of prices of related instruments such as variance swaps or barrier options.

6.1 General analysis

Before we discuss how to apply the above formalism of the forward characteristic function to pricing variance swaps under the described "3/2-power" clock change, let us first consider another possible approach. Let us assume that the Eq. (65) is valid, and let $s_t \equiv \ln\left(\frac{F_t}{F_0}\right)$ be the log price relative. Let:

$$\phi(u) \equiv E^{\mathbb{Q}}[e^{ius_T}|\mathcal{F}_T], \qquad u \in \mathbb{R}, \tag{70}$$

be the characteristic function of the log price relative. Using the law of iterated expectations, we have:

$$\phi(u) = E^{\mathbb{Q}}\left\{E^{\mathbb{Q}}[e^{ius_T}|\mathcal{F}_T^{\theta}]|\mathcal{F}_T\right\}, \tag{71}$$

where $\mathcal{F}_T^{\theta}$ indicates conditioning on the θ path over $[0,T]$. As it is shown in [25] the *conditional* characteristic function $E^{\mathbb{Q}}[e^{ius_T}|\mathcal{F}_T^{\theta}]$ depends on the particular θ path only through the sufficient statistic:

$$I_0 \equiv \int_0^T e^{\kappa\int_0^{t'}\theta(u)du}dt'. \tag{72}$$

In other words, if two θ paths $\{\theta_1(u), u \in [0,T]\}$ and $\{\theta_2(u), u \in [0,T]\}$ lead to the same value of I_0, then the value of the conditional characteristic function is the same. As a result, (71) implies that the unconditional characteristic function has the form:

$$\phi(u) = \int_0^{\infty} \psi(u, I_0)q(I_0)dI_0, \tag{73}$$

for all $u \in \mathbb{R}$, where $\psi(u, I_0)$ is the conditional characteristic function and $q(I_0)$ is the unknown risk-neutral density of I_0. From [25] the conditional characteristic function is given by:

$$\psi(u, I_0) = e^{iux} \frac{\Gamma(\tilde{\gamma} - \tilde{\alpha})}{\Gamma(\tilde{\gamma})} \left(\frac{2}{\epsilon^2 I_t v} \right)^{\tilde{\alpha}} M \left(\tilde{\alpha}; \tilde{\gamma}; \frac{-2}{\epsilon^2 I_t v} \right). \tag{74}$$

where $M(\tilde{\alpha}; \tilde{\gamma}; z)$ denotes the confluent hypergeometric function of the first kind, and:

$$\tilde{\alpha} \equiv -\left(\frac{1}{2} + \frac{\tilde{\kappa}}{\epsilon^2} \right) + \sqrt{\left(\frac{1}{2} + \frac{\tilde{\kappa}}{\epsilon^2} \right)^2 + \frac{u(u+i)}{\epsilon^2}},$$

$$\tilde{\gamma} \equiv 2 \left[\tilde{\alpha} + 1 + \frac{\tilde{\kappa}}{\epsilon^2} \right],$$

and where:

$$\tilde{\kappa} \equiv \kappa - \rho \epsilon i u.$$

In (73), we treat the LHS as a known function of u obtained from the market prices of European options of all strikes maturing at T. We think of the RHS as an integral transform of the unknown risk-neutral density $q(I_0)$. Since the kernel $\psi(u, I_0)$ is known for all $u \in \mathbb{R}$ and all $I_0 > 0$, one can theoretically invert for $q(I_0)$.

Now consider the problem of determining the Laplace transform of the risk-neutral density of the realized quadratic variation:

$$\mathcal{L}(\lambda) \equiv E^{\mathbb{Q}}[e^{-\lambda \int_0^T v_t dt} | \mathcal{F}_T], \qquad \lambda > 0. \tag{75}$$

Again, using the law of iterated expectations, we have:

$$\mathcal{L}(\lambda) = E^{\mathbb{Q}} \left\{ E^{\mathbb{Q}}[e^{-\lambda \int_0^T v_t dt} | \mathcal{F}_T^{\theta}] | \mathcal{F}_T \right\}, \tag{76}$$

where $\mathcal{F}_T^{\theta}$ again indicates conditioning on the θ path over $[0, T]$. As shown in [25] the conditional Laplace Transform $E^{\mathbb{Q}}[e^{-\lambda \int_0^T v_t dt} | \mathcal{F}_T^{\theta}]$ depends on the particular θ path only through the sufficient statistic I_0 defined in (72). As a result, (76) implies that the unconditional Laplace transform has the form:

$$\mathcal{L}(\lambda) = \int_0^{\infty} C^L(\lambda, I_0) q(I_0) dI_0, \tag{77}$$

for all $\lambda > 0$, where $C^L(\lambda, I_0)$ is the conditional Laplace transform of the risk-neutral PDF of the realized variance and where $q(I_0)$ is the now known risk-neutral density of I_0. The conditional Laplace transform of the risk-neutral density of the realized quadratic variation reads ([25])

$$C^L(\lambda, I_t) \equiv L(t, v) = \frac{\Gamma(\gamma - \alpha)}{\Gamma(\gamma)} \left(\frac{2}{\epsilon^2 I_t v} \right)^{\alpha} M \left(\alpha; \gamma; \frac{-2}{\epsilon^2 I_t v} \right), \tag{78}$$

where α and γ are defined after the Eq. (74), and recall that

$$I_t \equiv \int_t^T e^{\kappa \int_t^{t'} \theta(u) du} dt'. \tag{79}$$

Hence, by real inversion of this Laplace transform, the risk-neutral density of the realized quadratic variation can be obtained and hence realized volatility derivatives can be priced.

6.2 Closed-form solution for the variance swap

In this section we follow our algorithm that has been described earlier as applied to the affine clock change in order to derive a closed-form solution for the variance swap price under the stochastic "3/2-power" clock change. Let us consider Eq. (69), where now $\theta = \theta(t)$ is a known deterministic function of time. Again we consider the forward characteristic function of an arbitrary Lévy process with the characteristic exponent $\Psi_x(u)$ under the stochastic clock change determined by the "3/2-power" law. Similarly to Eq. (51)

$$\phi_{t_{i-1},t_i}(u) = e^{iu(r-q)\tau}\mathbb{E}_{\mathbb{Q}}\left[\mathcal{L}^u_{\mathbf{T}_\tau}(\Psi_x(u)) \;\middle|\; \nu_0\right] = e^{iu(r-q)\tau}\mathbb{E}_{\mathbb{Q}}\left[C^L\left(\Psi_x(u), I_{t_i}\right) \;\middle|\; \nu_0\right], \tag{80}$$

where

$$\begin{aligned} C^L(\lambda, I_{t_i}) &= \frac{\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}\left(\frac{2}{\epsilon^2 I_{t_i} v}\right)^{\alpha} M\left(\alpha;\gamma;\frac{-2}{\epsilon^2 I_{t_i} v}\right), \\ \alpha &= -\left(\frac{1}{2}+\frac{\kappa}{\epsilon^2}\right)+\sqrt{\left(\frac{1}{2}+\frac{\kappa}{\epsilon^2}\right)^2+2\frac{\Psi_x(u)}{\epsilon^2}}, \\ \gamma &\equiv 2\left[\alpha+1+\frac{\kappa}{\epsilon^2}\right], \\ I_{t_i} &\equiv \int_{t_{i-1}}^{t_i} e^{\kappa\int_{t_{i-1}}^{t'}\theta(u)du}dt' \end{aligned} \tag{81}$$

Now we make an assumption that $\kappa\theta(t)\tau \ll 1$ is a small parameter. This is a generalization of the assumption $\kappa\tau \ll 1$, that we made for the CIR clock change, for the case of the "3/2- power" model. Therefore, we expand the above expression in series on $\kappa\theta(t)\tau$ up to the linear terms.

First of all, expansion of $I_{t_{i-1}}$ reads

$$I_{t_i} = \tau + \kappa\theta(\tau)\tau^2 + O(\tau^2), \tag{82}$$

and therefore

$$z = \frac{2}{\epsilon^2 I_{t_{i-1}} v_t} \approx \frac{2}{\epsilon^2 \tau v_t} \tag{83}$$

As per [26] (13.5.1) an asymptotic expansion series for $M(\alpha;\gamma;z)$ at large $|z|$ reads

$$\begin{aligned} M(\alpha;\gamma;z) = {} & \frac{e^{i\pi\alpha}\Gamma(b)}{\Gamma(b-a)}z^{-\alpha}\left[\sum_{n=0}^{R-1}\frac{(\alpha)_n(1+\alpha-\gamma)_n}{n!}(-z)^{-n} + O\left(|z|^{-R}\right)\right] \\ & + \frac{e^z\Gamma(b)}{\Gamma(a)}z^{\alpha-\gamma}\left[\sum_{n=0}^{S-1}\frac{(\gamma-\alpha)_n(1-\alpha)_n}{n!}(-z)^{-n} + O\left(|z|^{-S}\right)\right], \quad -\frac{3}{2}\pi < argz < \frac{3}{2}\pi. \end{aligned} \tag{84}$$

We keep the first two terms in these series with $n = 0, 1$. Further, omitting a tedious algebra and remembering that $\Gamma(0) = \infty$ we find that

$$-\left.\frac{\partial^2\phi_{t_{i-1},t_i}(u)}{\partial u^2}\right|_{u=0} = (\Psi_x)''_u(0)\mathbb{E}_{\mathbb{Q}}[\tau v_{t_i}]. \tag{85}$$

Using this formula together with the Eq. (7) we obtain exactly the same result as for the CIR process Eq. (29), i.e.

$$\mathcal{Q}_N(s) = (\Psi_x)''_u(0)\frac{1}{T}\sum_{i=1}^N \mathbb{E}_{\mathbb{Q}}\left[\tau\nu_{i-1}|\nu_0\right] \approx (\Psi_x)''_u(0)\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{T}\int_0^T \nu_t dt \mid \nu_0\right] \equiv (\Psi_x)''_u(0)\mathbb{E}_{\mathbb{Q}}[V]. \tag{86}$$

The only difference is that now $\mathbb{E}_{\mathbb{Q}}[V]$ is computed using the "3/2-power" law, rather than the CIR process. This can be done by using a representation of the Laplace transform obtained in [25]. Indeed, we have

$$L(t,v) \equiv \mathbb{E}_{\mathbb{Q}}\left[e^{-\lambda \int_t^T v_u du} \middle| v_t = v \right], \qquad v \geq 0, t \in [0,T]$$

and thus

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[V] \equiv \mathbb{E}_{\mathbb{Q}}\left[\int_t^T v_u du \;\middle|\; v_t = v_0 \right] &= -\frac{\partial L(t,v)}{\partial \lambda}\Big|_{\lambda=0} \qquad (87) \\
&= \left[-\log\left(\frac{2}{\epsilon^2 I_T v_0}\right) + \frac{\Gamma'(2\nu)}{\Gamma(2\nu)} - 2M^{(0,1,0)}\left(0; 2\nu; -\frac{2}{\epsilon^2 I_T v_0}\right) - M^{(1,0,0)}\left(0; 2\nu; -\frac{2}{\epsilon^2 I_T v_0}\right)\right] \frac{\partial \alpha}{\partial \lambda}\Big|_{\lambda=0} \\
&= \left[-\log\left(\frac{2}{\epsilon^2 I_T v_0}\right) + \frac{\Gamma'(2\nu)}{\Gamma(2\nu)} - M^{(1,0,0)}\left(0; 2\nu; -\frac{2}{\epsilon^2 I_T v_0}\right)\right] \frac{2}{2\kappa + \epsilon^2},
\end{aligned}$$

where

$$\nu = 1 + \frac{\kappa}{\epsilon^2}, \qquad I_T \equiv \int_0^T e^{\kappa \int_0^{t'} \theta(u) du} dt',$$

$M^{(1,0,0)}(\alpha,\gamma,\zeta)$ is the derivative of $M(\alpha,\gamma,\zeta)$ on α, $M^{(0,1,0)}(\alpha,\gamma,\zeta)$ is the derivative of $M(\alpha,\gamma,\zeta)$ on γ, $\Gamma'(2\nu) \equiv d\Gamma(x)/dx|_{x=2\nu}$, and as follows from [26] (13.1.2) $M^{(0,1,0)}(0,\gamma,\zeta) = 0$.

As it can be easily validated, at short maturities when $T \to 0$ the integral $I_T \to 0$ as well, and from Eq. (84)

$$M^{(1,0,0)}(0,b,-\infty) \to \log\left(\frac{2}{\epsilon^2 I_T v_0}\right) - \frac{\Gamma'(2\nu)}{\Gamma(2\nu)}$$

Therefore, $\mathbb{E}_{\mathbb{Q}}[V] \to 0$ as expected, i.e. in this limit the equation Eq. (87) is consistent.

From a practitioner point of view computing the derivative of the confluent hypergeometric function on the first parameter could be kind of tricky. One possible way to eliminate this is to make use of the definition of the Kummer function given in [26]. By comparing the series expansion it could be verified that

$$M^{(1,0,0)}(0;\gamma;-z) = \sum_{i=1}^{\infty} \frac{z^i}{i(\gamma)_i} = -\left(\frac{z}{\gamma}\right) {}_2F_2[(1,1);(2,1+\gamma);-z], \qquad (88)$$

where ${}_2F_2(a_1,...,a_p,b_1,...,b_q,z)$ is the generalized hypergeometric function (*HypergeometricPFQ* in Mathematica notation, or *hypergeom* in Matlab).

Another approach could be as follows. Let us consider the hypergeometric equation (see, for instance, [25], Eq.225)

$$zh''(z) + (\gamma - z)h'(z) - \alpha h(z) = 0. \qquad (89)$$

According to [25] it has the solution

$$h(z) = \frac{\Gamma(\gamma-\alpha)}{\Gamma(\gamma)(-1)^{\alpha}} M(\alpha;\gamma;z) \qquad (90)$$

Let us differentiate the Eq. (89) on the parameter α, and then put $\alpha = 0$ to obtain

$$zw''(z) + (\gamma - z)w'(z) = 1, \qquad (91)$$

where $w(z) = \partial h(z)/\partial\alpha$, and we took into account that $h(z)|_{\alpha=0} = 1$. This equation has the following solution

$$\begin{aligned}
w(z) &= C_1 + C_2 \mathcal{I}_1(z) - \mathcal{I}_2(z), \qquad (92) \\
\mathcal{I}_1(z) &= \int_1^z \frac{e^t}{t^{\gamma}} dt, \qquad \mathcal{I}_2(z) = \int_1^z \frac{e^t}{t^{\gamma}} \Gamma(\gamma,t) dt
\end{aligned}$$

where $\Gamma(x,y)$ is the incomplete gamma function. Differentiating now Eq. (90) on α and comparing with the Eq. (92) gives

$$C_1 = -\frac{\Gamma'(\gamma)}{\Gamma(\gamma)} + M'_\alpha(0;\gamma;1), \qquad C_2 = \frac{\mathcal{I}_2(0) - M'_\alpha(0;\gamma;1)}{\mathcal{I}_1(0)} \tag{93}$$

Now from the Eq. (87) we obtain

$$\mathbb{E}_\mathbb{Q}[V] = \frac{2}{2\kappa+\epsilon^2}\left\{ -\log(z) + \frac{\Gamma'(2\nu)}{\Gamma(2\nu)} + \left(\frac{1}{2\nu}\right) {}_2F_2[(1,1);(2,1+\gamma);1]) \left[\frac{\mathcal{I}_1(z)}{\mathcal{I}_1(0)} - 1\right] + \mathcal{I}_2(z) - \mathcal{I}_2(0)\frac{\mathcal{I}_1(z)}{\mathcal{I}_1(0)}\right\}, \tag{94}$$

$$z \equiv \frac{2}{\epsilon^2 I_T v_0}$$

As in the Eq. (87) the long-term run coefficient $\theta(t)$ is an arbitrary function of time, it gives one a very nice opportunity to better calibrate this model to the real market data.

Another important observation is that, for instance, the CIR model for the stochastic time change is linear in drift. In other words, the SDE which governs the stochastic variance v_t has a drift term linear in v_t. Therefore, let us assume that the instantaneous variance V_t is mean reverting in general, i. e.

$$dV_t = k(\theta - V_t)dt + d\mathcal{M}_t \tag{95}$$

where $d\mathcal{M}_t$ is the increment at t of a martingale, e.g. $d\mathcal{M}_t = w(V_t,t)dW_t$. Then for any choice of $\mathcal{M}$, it is easy to give a closed form expression for the fair strike of the variance swap. Just note that

$$\mathbb{E}_\mathbb{Q}[dV_t] = d\mathbb{E}_\mathbb{Q}[V_t] = k(\theta - \mathbb{E}_\mathbb{Q}[V_t])dt \tag{96}$$

Hence if $\mu(t) = \mathbb{E}_\mathbb{Q}[V_t]$, then (96) implies the first order linear ODE $\mu'(t) = k(\theta - \mu(t))$. Solving this subject to $\mu(0) = V_0$ and integrating over t from 0 to T gives fair strike of the variance swap. Note that the answer is independent of how the volatility of V_t is specified.

In contrast, when the drift of V_t is nonlinear, e.g. quadratic then the answer depends on how the volatility of V_t is specified. Our "3/2-power" process gives one way to proceed.

7 Numerical experiments

As an numerical example first we determine a fair strike of the quadratic variation for three models.

Heston model. Considering the Heston model as a pure diffusion process (GBM with drift μ and volatility 1) under the CIR time change, the expression for the characteristic exponent of this process reads

$$\Psi_x(u) = -i\mu u + \frac{1}{2}\sigma^2 u^2, \tag{97}$$

therefore $\Psi''_x(u)|_{u=0} = \sigma^2$.

The Heston model has 5 free parameters $\kappa, \theta, \eta, \rho, v_0$ that can be obtained by calibrating the model to European option prices. In doing so one can use an FFT method as in Carr and Madan [9].

SSM. The second model is the stochastic skew model of Carr and Wu that has been briefly described in section 4. To complete the description of the model we specify two jump components J_t^L and J_t^R using the following specification for the Lévy density [11]

$$\mu^R(x) = \begin{cases} \lambda^R e^{-|x|/\nu_j^R} |x|^{-\alpha-1}, & x > 0, \\ 0, & x < 0. \end{cases} \qquad \mu^L(x) = \begin{cases} 0, & x > 0, \\ \lambda^L e^{-|x|/\nu_j^L} |x|^{-\alpha-1}, & x < 0. \end{cases} \tag{98}$$

so that the right-skewed jump component only allows up jumps and the left-skewed jump component only allows down jumps. In contrast to [11] for both type of jumps, we use different parameters $\lambda, \nu_j \in \mathbb{R}^+$. This specification has its origin in the CGMY model of Carr, Geman, Madan, and Yor [4]. The Lévy density of the CGMY specification follows an exponentially dampened power law. Depending on the magnitude of the power coefficient α the sample paths of the jump process can exhibit finite activity ($\alpha < 0$), infinite activity with finite variation ($0 < \alpha < 1$), or infinite variation ($1 < \alpha < 2$). Therefore, this parsimonious specification can capture a wide range of jump behaviors. Further we put $\alpha = -1$, so the jump specification becomes a finite-activity compound Poisson process with an exponential jump size distribution as in Kou [27].

For such Lévy density the characteristic exponent has the following form

$$\begin{aligned} \Psi_x^R(u) &= -iu\lambda^R \left[\frac{1}{1 - iu\nu_j^R} - \frac{\nu_j^R}{1 - \nu_j^R} \right] + \Psi_d^R(u) \\ \Psi_x^L(u) &= iu\lambda^L \left[\frac{1}{1 + iu\nu_j^L} - \frac{\nu_j^L}{1 + \nu_j^L} \right] + \Psi_d^L(u) \\ \Psi_d^k(u) &= \frac{1}{2}(\sigma^k)^2 (iu + u^2), \quad k = L, R, \end{aligned} \tag{99}$$

where $\Psi_d^k(u)$ is the characteristic exponent for the concavity adjusted diffusion component $\sigma W_t - \frac{1}{2}\sigma^2 t$.

Thus, form Eq. (99) we find that $(\Psi_x^k)''(0) \equiv (\sigma^k)^2 + 2\lambda^k n_j^k, \quad k = L, R$.

Overall, the SSM model has 16 free parameters $\kappa^k, \theta^k, \eta^k, \rho^k, v_0^k, \sigma^k, \lambda^k, \nu_j^k, \quad k = L, R$ that can be obtained by calibrating the model to European option prices, again using the FFT method.

NIG-CIR. The normal inverse Gaussian distribution is a mixture of normal and inverse Gaussian distributions. The density of a random variable that follows a NIG distribution $X \approx NIG(\alpha, \beta, \mu, \delta)$ is given by (see [28])

$$f_{NIG}(x; \alpha, \beta, \mu, \delta) = \frac{\delta \alpha e^{\delta\gamma + \beta(x-\mu)}}{\pi \sqrt{d^2 + (x-\mu)^2}} K_1 \left(\alpha \sqrt{\delta^2 + (x-\mu)^2} \right), \tag{100}$$

where $K_1(w)$ is the modified Bessel function of the third kind.

As a member of the family of generalized hyperbolic distribution, the NIG distribution is infinitely divisible and thus generates a Lévy process $(L_t)_{t>0}$. For an increment of length s , the NIG Lévy process satisfies

$$L_{t+s} - L_t \approx \mathrm{NIG}(\alpha, \beta, \mu s, \delta s) \tag{101}$$

Combined with the CIR clock change it produces a NIG-CIR model. The possible values of the parameters are $\alpha > 0, \delta > 0, \beta < |\alpha|$, while μ can be any real number.

Below for convenience we use transformed variables, namely:

$$\Theta \equiv \beta/\delta, \quad \nu \equiv \delta \sqrt{\alpha^2 - \beta^2}$$

The characteristic exponent of the NIG model reads

$$\Psi_x(u) = iu\mu + \delta \left[\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu)^2} \right] \tag{102}$$

Calibration. For all the tests given below we first retrieved the vanilla option price data from `http://finance.yahoo.com/`. If the option was American, we computed its implied volatility and used this value to find a corresponding European vanilla option price. We used just the OTM puts and calls. Then to calibrate the model we minimized a nonlinear least-square functional

$$\min_{p_1 \dots p_N} \sum_{i=1}^{L} w_i (V_{i,market} - V_{i,model})^2, \tag{103}$$

where $p_1 ... p_N$ are the best fit parameters of the model to be found, $V_{i,market}, i = 1, L$ is the set of market data, $V_{i,model}, i = 1, L$ are the corresponding theoretical values, and $w_i, i = 1, L$ are the weights. If the characteristic function of the model is known in the closed form we used the Carr-Madan procedure to find the option price via FFT. The weights were defined as by $w_i = 1/(NS)^q$, where NS is the normalized strike, $NS = \log(K/F)/(\sigma_{ATM}\sqrt{T})$, K is the option strike, F is the forward price, σ_{ATM} is the option ATM volatility, q is some constant, that in the below tests was chosen $q = 2$. Therefore, the option prices closer to the ATM acquired a bigger weight. The interest rates were averaged through the time period used in the calibration routine.

The Eq. (103) was solved using differential evolution - a global optimization method which belongs to the class of genetic algorithms. The set of parameters to be calibrated was chosen appropriately. For instance, as it is known, for the Heston model only 4 of the 5 parameters could be independently obtained by calibration.

Results. We use these three models to compute the fair value of the quadratic variation contract on SMP500 and Google on August 14, 2006. Parameters of the models were obtained by calibrating them to the 480 available European option prices. We found the following values of the calibrated parameters (see Tables 1-3)

κ	θ	η	v_0	ρ
1.572	0.038	0.504	0.019	-0.699

Table 1: Calibrated parameters of the Heston model

κ_L	θ_L	η_L	v_{0L}	ρ_L	σ_L	λ_L	ν_L
1.2916	0.6515	2.1152	0.3366	-0.9998	0.2077	0.02396	1.8455
κ_R	θ_R	η_R	v_{0R}	ρ_R	σ_R	λ_R	ν_R
6.7486	1.999	0.0004	0.0002	0.4049	0.0734	0.0029	0.5864

Table 2: Calibrated parameters of the SSM model

κ	θ	η	v_0	ρ	δ	ν	Θ	μ
2.855	0.093	0.787	0.057	-0.987	0.897	7.533	-1.285	0.482

Table 3: Calibrated parameters of the NIGCIR model

It is interesting to see whether the term structure of the variance swap prices computed using these models and the values of the parameters obtained by calibration is able to replicate the market data. To remind, all the model's parameter do not depend on time. Therefore, we compared the fair swap price obtained in such a way with that given by the log contract for SPX (Fig. 1) and Google (Fig. 2). The log contract data were obtained from Bloomberg. As it could be seen usage of the SSM model slightly improves an agreement with the log contract as compared with the Heston model. But, nevertheless, the difference is substantial, especially at large maturities.

8 Conclusion

In this paper we investigated variance and volatility swaps and options on these instruments under discrete observations. We proposed a new asymptotic method which aims to obtain a closed-form expression for the fair price of these instruments, if the underlying process is modeled by a Lévy process with stochastic time change. This is done in two cases.

The first one is when the stochastic time change process belongs to the class of affine processes. We began with the case when the annualized time between the observations is relatively small and considered the activity rate models with a rather general jump specification proposed by Carr and Wu [11]. Using our method we proved that under this specification the annualized quadratic variation of the Lévy process with stochastic time determined by a **pure diffusion process** is given by the annualized realized variance times a constant coefficient ξ. This coefficient is determined via derivatives of the characteristic function of the underlying Lévy process. The examples given in the paper consider the CIR clock change for the Black-Scholes model (which is actually the Heston model), NIG model and SSM model. We also proved the Theorem that the annualized quadratic variation of the Lévy process under stochastic time determined by a **jump-diffusion process** is also given by a product of the annualized realized variance and a constant coefficient ξ **plus** some constant η which is determined via derivatives of the characteristic function of the underlying Lévy process and jump integrals of the time change process. We further managed to extend our results by investigating a more general case when discrete observations of the underlying spot price occur over a bigger time interval. We showed in the Theorem 3 that in this case the formulae for the price of the quadratic variation swap acquire two extra terms. The first one $p_0(\tau)$ is a function of time between observations τ and is determined by a particular model of the underlying Lévy process. The last term $p_2(\tau)\mathbb{E}_{\mathbb{Q}}[V^2]$ is proportional to the square of variance and is some kind of convexity adjustment. These two extra terms appears only in the second order approximation on the time interval between the observations τ. Therefore, in the case of rare discrete observations the standard log-contract price (which is, in fact, an expectation of the realized variance) is no longer valid. For the particular case of the CIR time change the second term p_2 vanishes even in the second order of approximation in τ.

The second case considered in the paper is when the stochastic time change follows the so-called "3/2 power" process which is not affine. For this model the closed-form expression for the fair price of the variance and volatility swaps was also obtained in the closed-form.

The above results could be helpful because they allow fast pricing of the above instruments under rather complicated models, which in turn proved to be able to catch many characteristics of the underlying process. However, given numerical examples and comparison with the market data indicate that even these complicated models (at least, these particular three models used in our tests) are not able to capture the term-structure of the variance swaps. One possible way of achieving that is a known approach of considering the long term coefficient of the mean-reverting part of the variance process to be stochastic as well. So for the future it would be interesting to try applying our approach to this kind of models.

Appendix: Typical CBOE contract on variance swaps

S&P 500 3-month Variance Contracts CBOE S&P 500 3-month Variance Futures are based on the realized, or historical, variance of the S&P 500 Index. CBOE S&P 500 3-month Variance Futures are quoted in terms of variance points, which are defined as realized variance multiplied by 10,000. One variance point is worth $50. For example, a variance calculation of 0.06335 would have a corresponding price quotation in variance points of 633.50, and a contract size of $31,675.00 (633.50 x $50).

The Final Settlement Value for CBOE S&P 500 3-month Variance Futures is calculated using continuously compounded daily S&P 500 returns over a three-month period, assuming a mean daily price return of zero. A "continuously compounded" daily return (Ri) is calculated from two reference values, an initial value P_i and a final value P_{i+1}, using the following formula:

$$R_i = \ln\left(\frac{P_{i+1}}{P_i}\right)$$

Daily returns are accumulated over a three-month period, and then used in a standardized formula to calculate three-month variance. This three-month value is then annualized assuming 252 business days per year:

$$\frac{252}{N_e - 1}\sum_{i=1}^{N_a - 1} R_i^2.$$

Here N_e is the number of expected S&P 500 values needed to calculate daily returns during the three-month period. The total number of daily returns expected during the three-month period is $N_e - 1$. N_a is the actual number of S&P 500 values used to calculate daily returns during the three-month period. Generally, the actual number of S&P 500 values will equal the expected number of S&P 500 values. However, if one or more "market disruption events" occurs during the three-month period, the actual number of S&P 500 values will be less than the expected number of S&P 500 values by an amount equal to the number of market disruption events that occurred during the three-month period. The total number of actual daily returns during the three-month period is $N_a - 1$.

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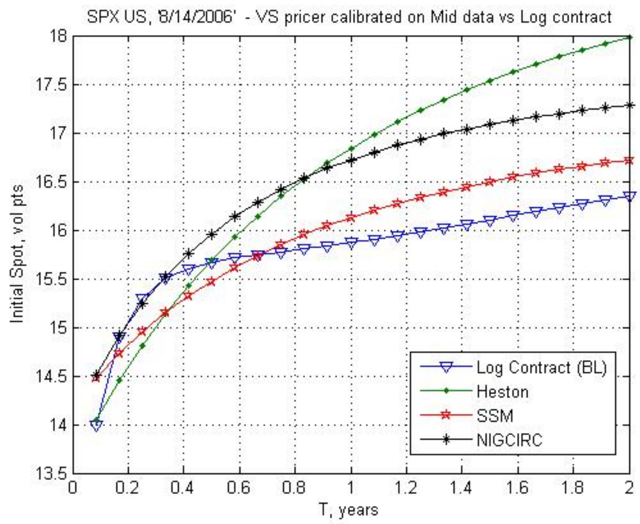

Figure 1: Fair strike of SPX in Heston, NIGCIR and SSM models. Comparison with a log contract (as per Bloomberg).

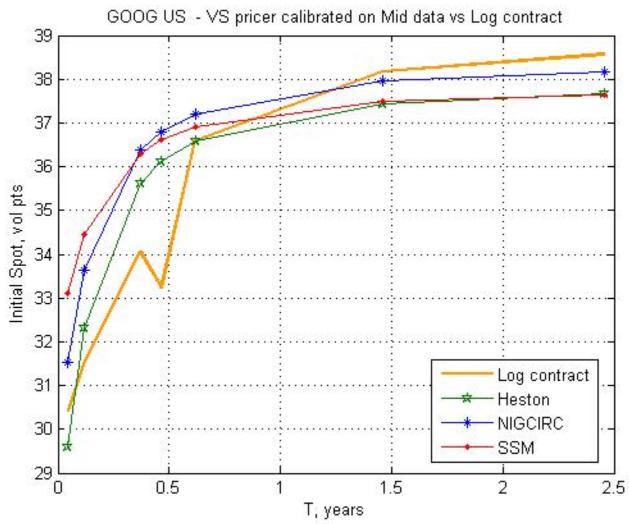

Figure 2: Same for Google