Using pseudo-parabolic equations for option pricing in jump diffusion models

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Outline

In mathematical finance a popular approach for pricing options under some Lévy model is to consider underlying that follows a Poisson jump diffusion process. As it is well known this results in a partial integro-differential equation (PIDE) that usually does not allow an analytical solution while numerical solution brings some problems. In this work we elaborate a new approach on how to transform the PIDE to some class of so-called pseudo-parabolic equations which are known in mathematics but are relatively new for mathematical finance. As an example we discuss several jump-diffusion models which Lévy measure allows such a transformation.

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Existing approaches - cont.

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Alternative methods for real α

1. Suppose we consider GTSP/KoBoL/SSM class of models. We will transform the corresponding PIDE to a fractional PDE. Fractional PDEs for Lévy processes with finite variation were obtained by Boyarchenko and Levendorsky (2002) and later by Cartea (2007) using a characteristic function technique.
2. We derive it in all cases including processes with infinite variation using a different technique - shift operators.

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3. and a pure jump PIDE (could be always obtained by using splitting) reads (positive jumps, but negative - by analogy)

$$\frac{\partial}{\partial \tau} C(x,\tau) = \mathcal{B}_1 C(x,\tau)$$

$$\mathcal{B}_1 \equiv \int_0^\infty \left[\exp\left(y\frac{\partial}{\partial x}\right) - 1 - (e^y - 1)\frac{\partial}{\partial x}\right] \lambda_R \frac{e^{-\nu_R|y|}}{|y|^{1+\alpha_R}} dy \tag{2}$$

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Main theorem

Theorem (1)

The PIDE

$$\frac{\partial}{\partial \tau} C(x,\tau) = \int_0^\infty \left[C(x+y,\tau) - C(x,\tau) - \frac{\partial}{\partial x} C(x,\tau)(e^y - 1) \right] \lambda_R \frac{e^{-\nu_R |y|}}{|y|^{1+\alpha_R}} dy \tag{3}$$

is equivalent to the fractional PDE

$$\frac{\partial}{\partial \tau} C(x,\tau) = \lambda_R \Gamma(-\alpha_R) \left\{ \left(\nu_R - \frac{\partial}{\partial x} \right)^{\alpha_R} - \nu_R^{\alpha_R} + \left[\nu_R^{\alpha_R} - (\nu_R - 1)^{\alpha_R} \right] \frac{\partial}{\partial x} \right\} C(x,\tau),$$
$$\mathbb{R}(\alpha_R) < 2, \; \mathbb{R}(\nu_R - \partial/\partial x) > 0, \; \mathbb{R}(\nu_R) > 1. \tag{4}$$

In special cases this equation changes to

$$\frac{\partial}{\partial \tau} C(x,\tau) = \lambda_R \left\{ \log(\nu_R) - \log\left(\nu_R - \frac{\partial}{\partial x}\right) + \log\left(\frac{\nu_R - 1}{\nu_R}\right) \frac{\partial}{\partial x} \right\} C(x,\tau) \tag{5}$$
$$\alpha_R = 0, \mathbb{R}(\nu_R - \partial/\partial x) > 0, \mathbb{R}(\nu_R) > 1,$$

and

$$\frac{\partial}{\partial \tau} C(x,\tau) = \lambda_R \Big\{ -\nu_R \log \nu_R + (\nu_R - \frac{\partial}{\partial x}) \log\left(\nu_R - \frac{\partial}{\partial x}\right) + [\nu_R \log \nu_R - (\nu_R - 1)\log(\nu_R - 1)] \frac{\partial}{\partial x} \Big\} C(x,\tau)$$
$$\alpha_R = 1, \mathbb{R}(\partial/\partial x) < 0, \mathbb{R}(\nu_R) > 1,$$

Main theorem - cont.

Theorem (2)

The PIDE

$$\frac{\partial}{\partial \tau} C(x,\tau) = \int_{-\infty}^{0} \left[C(x+y,\tau) - C(x,\tau) - \frac{\partial}{\partial x} C(x,\tau)(e^y - 1) \right] \lambda_L \frac{e^{-\nu_L |y|}}{|y|^{1+\alpha_L}} dy \tag{6}$$

is equivalent to the fractional PDE

$$\frac{\partial}{\partial \tau} C(x,\tau) = \lambda_L \Gamma(-\alpha_L) \left\{ \left(\nu_L + \frac{\partial}{\partial x} \right)^{\alpha_L} - \nu_L^{\alpha_L} + \left[\nu_L^{\alpha_L} - (\nu_L + 1)^{\alpha_L} \right] \frac{\partial}{\partial x} \right\} C(x,\tau),$$
$$\mathbb{R}(\alpha_L) < 2,\ \mathbb{R}(\nu_L + \partial/\partial x) > 0,\ \mathbb{R}(\nu_L) > 0. \tag{7}$$

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$$\alpha_L = 0,\ \mathbb{R}(\nu_L + \partial/\partial x) > 0,\ \mathbb{R}(\nu_L) > 0,$$

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$$\frac{\partial}{\partial \tau} C(x,\tau) = \lambda_L \Big\{ -\nu_L \log \nu_L + [\nu_L \log \nu_L - (\nu_L + 1)\log(\nu_L + 1)] \frac{\partial}{\partial x} + (\nu_L + \frac{\partial}{\partial x}) \log\left(\nu_L + \frac{\partial}{\partial x} \right) \Big\} C(x,\tau)$$
$$\alpha_R = 1,\ \mathbb{R}(\partial/\partial x) < 0,\ \mathbb{R}(\nu_L) > 0,$$

Solution

1. Remember that, for instance, the first equation could be represented in the form

$$\frac{\partial}{\partial t} C(x,t) = -\mathcal{B} C(x,t) \tag{9}$$

2. This equation can be formally solved analytically to give

$$C(x,t) = e^{\mathcal{B}(T-t)} C(x,T), \tag{10}$$

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3. Suppose that the whole time space is uniformly divided into N steps, so the time step $\theta = T/N$ is known. Assuming that the solution at time step $k, 0 \le k < N$ is known and we go backward in time, we could rewrite the above as
$$C^{k+1}(x) = e^{\mathcal{B}\theta} C^k(x), \tag{11}$$

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Padé approximations

1. First, consider the case $m = 1$. A symmetric Padé approximation of the order $(1, 1)$ for the exponential operator is

$$e^{\mathcal{B}\theta} = \frac{1 + \mathcal{B}\theta/2}{1 - \mathcal{B}\theta/2} \tag{12}$$

so we obtain a Crank-Nicolson scheme

$$\left(1 - \frac{1}{2}\mathcal{B}\theta\right) C^{k+1}(x) = \left(1 + \frac{1}{2}\mathcal{B}\theta\right) C^{k}(x). \tag{13}$$

2. The case $m = 2$ could be achieved either by using symmetric (2,2) or diagonal (1,2) Padé approximations of the operator exponent. The (1,2) Padé approximation reads

$$e^{\mathcal{B}\theta} = \frac{1 + \mathcal{B}\theta/3}{1 - 2\mathcal{B}\theta/3 + \mathcal{B}^2\theta^2/6}, \tag{14}$$

and the corresponding finite difference scheme for the solution of the Eq. (11) is

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Approximation in space - finite variation

1. Exclude the last compensator term in the PIDE - move it to a differential part.
2. Use a one-sided forward approximation of the first derivative which is a part of the operator $\left(\nu_R - \frac{\partial}{\partial x}\right)^{\alpha_R}$. Define $h = (x_{max} - x_{min})/N$ to be the grid step in the x-direction, N is the total number of steps, x_{min} and x_{max} are the left and right boundaries of the grid. Also define $c_i^k = C^k(x_i)$. To make our method to be of the second order in x we use the following numerical approximation

$$\frac{\partial C^k(x)}{\partial x} = \frac{-C_{i+2}^k + 4C_{i+1}^k - 3C_i^k}{2h} + O(h^2) \quad (16)$$

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4. To summarize these results let us represent them in Table 1,2 where F means the forward approximation, B means the backward approximation and S means the shifted approximation.

α_R, α_L	Operator	
	$\mathbb{B}^-$	$\mathbb{B}^+$
$\alpha < 0$	F	B
$0 < \alpha < 1$	F	B

Table: Finite difference approximation of the operators at $\alpha_R < 1$, $\alpha_L < 1$.

	Operator	
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1 part	SF	SB
2 part	F	B

Table: Finite difference approximation of the operators at $1 < \alpha_R < 2$ and $1 < \alpha_L < 2$.

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1. Special cases $\alpha_r = 0, 1, \alpha_l = 0, 1$ are not considered in Cartea (2007). In Boyarchenko & Levendorsky (2002) a corresponding characteristic function of the KoBoL process is obtained in these cases as well, but they did not consider the fractional PDE and its numerical solution.
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3. As it is known from recent papers Abu-Saman (2007), Meerschaert and Tadjeran (2004, 2006), Sousa (2008), a standard Grunwald-Letnikov approximation leads to unconditionally unstable schemes. To improve this a shifted Grunwald-Letnikov approximation was proposed which allows construction of the unconditionally stable scheme of the first order in space. Here we use a different approach to derive the unconditionally stable scheme of higher order.

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2. In Cartea (2007) a Crank-Nicolson type numerical scheme was proposed to solve the obtained FPDE which is of the first order in space. Here we derived high-order schemes in both time and space.
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4. We show that despite it is a common practice to integrate out all Lévy compensators in the integral term when one considers jumps with finite activity and finite variation, this breaks the stability of the scheme, at least for the fractional PDE. Therefore, in order to construct the unconditionally stable scheme one must keep the other terms under the integrals. To resolve this in Cartea (2007) the authors were compelled to change their definition of the fractional derivative.

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The end

Thank you!