Pricing options with VG model using FFT

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- Consider alternative methods - Lewis-wise and Black-Scholes-wise and show that they seem to work fine for any value of the VG parameters.

- Give test examples to demonstrate efficiency of these methods and discuss convergency and accuracy of all methods.
Brief overview of the VG model
VG model

Proposed by Madan and Seneta (1990) to describe stock price dynamics instead of the Brownian motion in the original Black-Scholes model.

Two new parameters: $\theta$ skewness and $\nu$ kurtosis are introduced in order to describe asymmetry and fat tails of real life distributions.

The VG process is defined by evaluating Brownian motion with drift at a random time specified by gamma process.

\[
\ln F_t = \ln F_0 + X_t + \omega t, \tag{2}
\]

where

\[
X_t = \theta \gamma_t(1, \nu) + \sigma W_{\gamma_t(1, \nu)}, \tag{3}
\]
and $\gamma_t(1, \nu)$ is a Gamma process playing the role of time in this case with unit mean rate and density function given by

$$f_{\gamma_t(1, \nu)}(x) = \frac{x_{\nu}^{t_{\nu} - 1}e^{-\frac{x}{\nu}}}{\nu_{\nu}^{t_{\nu}}\Gamma\left(\frac{t}{\nu}\right)}. \quad (4)$$

The probability density function for the VG process may be written as

$$h_t(x) = \int_0^\infty \frac{dg}{\sqrt{2\pi g}} \exp \left[ -\frac{(x - \theta g)^2}{2\sigma^2 g} \right] \frac{g_{\nu}^{t_{\nu} - 1}e^{-\frac{g}{\nu}}}{\nu_{\nu}^{t_{\nu}}\Gamma\left(\frac{t}{\nu}\right)} \quad (5)$$

or after integration over $g$

$$h_t(x) = \frac{2e^{\theta x/\sigma^2}}{\sqrt{2\pi\sigma\nu_{\nu}^{t_{\nu}}}\Gamma(t/\nu)} \left(\frac{x^2}{\theta^2 + \frac{2\sigma^2}{\nu}}\right)^{t_{\nu} - \frac{1}{4}} K_{\frac{t}{\nu} - \frac{1}{2}} \left(\frac{1}{\sigma^2}\sqrt{x^2 \left(\theta^2 + \frac{2\sigma^2}{\nu}\right)}\right), \quad (6)$$
where \( K \) is the modified Bessel function of the second kind. The characteristic function \( \phi_{\gamma_t(1,\nu)}(u) \) for the VG process has a remarkably simple form

\[
\phi_t(u) \equiv \left\langle e^{iux} \right\rangle = \int_0^\infty h_t(x) e^{iux} \, dx = \frac{1}{(1 - i\theta \nu u + \frac{1}{2} \sigma^2 \nu u^2)^{\frac{t}{\nu}}}. \tag{7}
\]

Now, to prevent arbitrage, we need \( F_t \) to be a martingale, and, since \( F_t \) is already an independent increment process, all we need is

\[
\mathbb{E}[F_t] = F_0, \tag{8}
\]

This tells us that

\[
\omega = -\frac{\ln \phi_{X_t}(-i)}{t} = -\frac{t}{\nu} \ln \left( 1 - \theta \nu - \frac{1}{2} \sigma^2 \nu \right) = \frac{1}{\nu} \ln \left( 1 - \theta \nu - \frac{1}{2} \sigma^2 \nu \right). \tag{9}
\]
VG model (continue)

From the definition of \( \omega \) above, in order to have a risk neutral measure for VG model, its parameters must obey an inequality:

\[
\frac{1}{\nu} > \theta + \frac{\sigma^2}{2}.
\]  \( \text{(10)} \)

Accordingly, the characteristic function of the \( x_T \equiv \log S_T \) VG process is

\[
\phi(u) = \frac{\left[ S_0 e^{(r-q+\omega)T} \right]^iu}{\left( 1 - i\theta \nu u + \frac{1}{2} \sigma^2 \nu u^2 \right)^{\frac{T}{\nu}}}. \]  \( \text{(11)} \)
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$$
\prod_{j=1}^{n} h_{\tau_j}(x_j),
$$

(14)

where the PDF $h_{\tau_j}(x_j)$ were given above, and $x_j$ are observed returns per time $\tau_j$, i.e. $x_j = \log(S_j/S_{j-1})$. 

VG model (continue)

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- In particular we have to find the values of the parameters $\theta^*, \nu^*$ and $\sigma^*$ such that the following expression is maximized:

$$\prod_{j=1}^{n} h_{\tau_j}(x_j),$$  \hspace{1cm} (15)

where the PDF $h_{\tau_j}(x_j)$ were given above, and $x_j$ are observed returns per time $\tau_j$, i.e. $x_j = \log(S_j/S_{j-1})$.

- Note that risk neutral parameters $\theta, \nu, \sigma$ do not have to be equal to their statistical counterparts.
Pricing European option
European option value

The value of European option on a stock when the risk neutral dynamics is given by VG is

\[ V = \exp(-rT) \int_{-\infty}^{\infty} h_T (x - (r - q + \omega)T) W(e^x)dx, \quad (16) \]

where \( T \) is time until expiration, \( q \) is continuous dividend and \( W(e^x) \) is payoff function that has the following form

\[ W(e^x) = (S_0e^x - K)^+ - \text{call}, \quad W(e^x) = (K - S_0e^x)^+ - \text{put}. \quad (17) \]

Direct calculation allows us to derive the put-call parity relation identical to Black-Scholes case

\[ C = S_0e^{-qT} - Ke^{-rT} + P. \quad (18) \]
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- Konikov and Madan in (“Variance gamma model: Gamma weighted Black-Scholes implementation”, Technical report, Bloomberg L.P., 2004) This method leads to a weighted sum of the BS formulae while has not been implemented yet.
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- Monte-Carlo methods - slow.
Carr-Madan’s FFT approach and VG

Once the characteristic function \( \phi(u, t) = \mathbb{E}(e^{iuX_t}) \), where \( X_t = \log(S_t) \), is available, then the vanilla call option can be priced using Carr-Madan’s FFT formula:

\[
C(K, T) = \frac{e^{-\alpha \log(K)}}{\pi} \int_0^\infty \text{Re} \left[ e^{-iv \log(K)} \omega(v) \right] dv, \quad (19)
\]

where

\[
\omega(v) = \frac{e^{-rT} \phi(v - (\alpha + 1)i, T)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v} \quad (20)
\]

The integral in the first equation can be computed using FFT, and as a result we get call option prices for a variety of strikes. The put option values can just be constructed from Put-Call parity.

Parameter \( \alpha \) must be positive. Usually \( \alpha = 3 \) works well for various models. It is important that the denominator has only imaginary roots while integration is provided along real \( v \). Thus, the integrand \( \omega(v) \) is well-behaved.
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(21)

where

$$\omega(v) = \frac{e^{-rT} \phi(v - (\alpha + 1)i, T)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}$$

(22)

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\]

where

\[
\omega(v) = \frac{e^{-rT} \phi(v - (\alpha + 1)i, T)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}. \tag{24}
\]

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Figure 2: European option values in VG model at $T = 0.02\text{yrs}$, $K = 90$, $\sigma = 0.01$ obtained with FRFT.

Figure 3: European option values in VG model at $T = 0.02\text{yrs}$, $K = 90$, $\sigma = 0.01$ obtained with the adaptive integration.
**Figure 4:** European option values in VG model at $T = 0.02\text{yrs}$, $K = 90$, $\sigma = 0.01$ obtained with FFT.

**Figure 5:** European option values in VG model at $T = 1.0\text{yrs}$, $K = 90$, $\sigma = 1.0$ obtained with the FFT.
Figure 6: European option values in VG model at $T = 1.0\text{ yrs}$, $K = 90$, $\sigma = 1.0$ obtained with the FRFT.

Figure 7: European option values in VG model at $T = 1.0\text{ yrs}$, $K = 90$, $\sigma = 1.0$ obtained with the adaptive integration.
European call option value in the Carr-Madan method

\[
C(K, T) \propto e^{-\alpha \log(K) - rT} \frac{1}{\pi} \int_0^\infty \Re(\Psi(v)) \, dv
\]

(25)

\[
\Psi(v) \equiv \frac{e^{-iv \log(K)}}{\left[\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v \right] \left(1 - i\theta \nu u + \sigma^2 \nu u^2 / 2\right)^{\frac{1}{v}}},
\]

where \(u \equiv v - (\alpha + 1)i\). At small \(T\) the denominator has no real roots. To understand what happens at larger maturities, let us put
\(T = 0.8, \nu = 0.1, \alpha = 3, \sigma = 1\) and see how the denominator behaves as a function of \(v\) and \(\Theta\).
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Carr and Madan’s condition to keep the characteristic function to be finite

\[
\alpha < \sqrt{\frac{2}{\nu \sigma^2} + \frac{\Theta^2}{\sigma^4} - \frac{\Theta}{\sigma^2} - 1}. \tag{28}
\]
Investigation (continue)

Figure 8: Denominator of $\Psi(v)$ at $T = 0.8, \nu = 0.1, \alpha = 3, \sigma = 1$ as a function of $v$ and $\Theta$.

Figure 9: Denominator of $\Psi(v)$ at $T = 0.8, \nu = 0.1, \alpha = 3, v = 0$ as a function of $\sigma$ and $\Theta$. 
Lewis regularization
Lewis method

Alan Lewis (2001) notes that a general integral representation of the European call option value with a vanilla payoff is

\[ C_T(x_0, K) = e^{-rT} \int_{-\infty}^{\infty} (e^x - K)^+ Q(x, x_0, T) dx, \] (29)

where \( x = \log S_T \) is a stock price that under a pricing measure evolves as \( S_T = S_0 \exp[(r - q)T + X_T] \), and \( X_T \) is some Levy process satisfying \( \mathbb{E}[\exp(iuX_T)] = 1 \), and \( Q \) is the density of the log-return distribution \( x \).

The central point of the Lewis’s work is to represent this equation as a convolution integral and then apply a Parseval identity

\[ \int_{-\infty}^{\infty} f(x)g(x_0 - x)dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux_0} \hat{f}(u)\hat{g}(u)du, \] (30)

where the hat over function denotes its Fourier transform.
The idea behind this formula is that the Fourier transform of a transition probability density for a Levy process to reach $X_t = x$ after the elapse of time $t$ is a well-known characteristic function. For Levy processes it is $\phi_t(u) = E[\exp(iuX_t)], u \in \mathbb{R}$, and typically has an analytic extension (a generalized Fourier transform) $u \rightarrow z \in \mathbb{C}$, regular in some strip $S_X$ parallel to the real $z$-axis.
Lewis method (continue)

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Suppose that the generalized Fourier transform of the payoff function

$$\hat{w}(z) = \int_{-\infty}^{\infty} e^{izx} (e^x - K)^+ dx$$

and $\phi_t(z)$ both exist, the call option value is

$$C_T(x_0, K) = e^{-rT} \mathbb{E} \left[ (e^X - K)^+ \right] = \frac{1}{2\pi} \mathbb{E} \left[ \int_{i\mu-\infty}^{i\mu+\infty} e^{-izT} \hat{w}(z) dz \right]$$

$$= \frac{e^{-rT}}{2\pi} \mathbb{E} \left[ \int_{i\mu-\infty}^{i\mu+\infty} e^{-iz[x_0+(r-q+\omega)T]} e^{-izX_T} \hat{w}(z) dz \right]$$

$$= \frac{e^{-rT}}{2\pi} \int_{i\mu-\infty}^{i\mu+\infty} e^{-izY} \phi_{X_T}(-z) \hat{w}(z) dz.$$

Here $Y = x_0 + (r - q + \omega)T$, $\mu \equiv \text{Im } z$. This is a formal derivation which becomes a valid proof if all the integrals exist.
The Fourier transform of the vanilla payoff can be easily found by a direct integration

\[
\hat{w}(z) = \int_{-\infty}^{\infty} e^{izx} (e^x - K)^+ dx = -\frac{K^{iz+1}}{z^2 - iz}, \quad \text{Im}z > 1. \tag{33}
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Lewis method (continue)

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Note that if \( z \) were real, this regular Fourier transform would not exist. As shown by Lewis, payoff transforms \( \hat{w}(z) \) for typical claims exist and are regular in their own strips \( S_w \) in the complex \( z \)-plane, just like characteristic functions.
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\[
C(S, K, T) = -\frac{Ke^{-rT}}{2\pi} \int_{i\mu - \infty}^{i\mu + \infty} e^{-izk} \phi_{X_T}(-z) \frac{dz}{z^2 - iz}, \quad \mu \in S_V, \quad (38)
\]

and \( k = \log(S/K) + (r - q + \omega)T \).
Lewis method (continue)

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\]

and \( k = \log(S/K) + (r - q + \omega)T \).
The characteristic function of the VG process is defined in the strip 
\( \beta - \gamma < \text{Im} \ z < \beta + \gamma \), where

\[
\beta = \frac{\Theta}{\sigma^2}, \quad \gamma = \sqrt{\frac{2}{\nu \sigma^2} + \frac{\Theta^2}{\sigma^4} + 2(\text{Re} \, z)^2}.
\] (41)

This condition can be relaxed by assuming \( \text{Re} \, z = 0 \). Accordingly, \( \phi(-z) \) is defined in the strip \( \gamma - \beta > \text{Im} \ z > -\beta - \gamma \).
Lewis method - existence

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is defined in the strip \( \gamma - \beta > \text{Im} \ z > -\beta - \gamma \). Choose \( \text{Im} \ z \) in the form

\[
\mu \equiv \text{Im} \ z = \sqrt{1 + \frac{2\Theta}{\sigma^2} + \frac{\Theta^2}{\sigma^4} - \frac{\Theta}{\sigma^2}}.
\] (44)

It is easy to see that \( \mu \) defined in such a way obeys the inequality
\( \mu < \gamma - \beta \). On the other hand, \( \mu \geq 1 \) at any value of \( \Theta \) and positive
volatilities \( \sigma \), and the equality is reached when \( \Theta = 0 \). It means, that \( \text{Im} \ z = \mu \)
lies in the strip \( S^*_X \) as well as in the strip \( S_w \), i.e. \( \mu \in S_V = S^*_X \cap S_w \).
The integrand in Eq. (39) is regular throughout $S^*_X$ except for simple poles at $z = 0$ and $z = i$. The pole at $z = 0$ has a residue $-Ke^{-rT}i/(2\pi)$, and the pole at $z = i$ has a residue $Se^{-qT}i/(2\pi)$. 

The integrand in Eq.(39) is regular throughout $S_X^*$ except for simple poles at $z = 0$ and $z = i$. The pole at $z = 0$ has a residue $-K e^{-rT i}/(2\pi)$, and the pole at $z = i$ has a residue $S e^{-qT i}/(2\pi)$.

Strip $S_X^*$ is defined by the condition $\gamma - \beta > \text{Im} z > -\beta - \gamma$, where $\gamma - \beta > 1$, and $-\beta - \gamma < 0$. We can move the integration contour to $\mu_1 \in (0, 1)$ and use the residue theorem.
**Lewis method - contour integration**

The integrand in Eq.(39) is regular throughout $S_X^*$ except for simple poles at $z = 0$ and $z = i$. The pole at $z = 0$ has a residue $-K e^{-rT} i/(2\pi)$, and the pole at $z = i$ has a residue $S e^{-qT} i/(2\pi)$.

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First alternative formula

$$C(S, K, T) = S e^{-qT} - \frac{K e^{-rT}}{2\pi} \int_{i\mu_1 - \infty}^{i\mu_1 + \infty} e^{-izk} \phi_X(\zeta) \frac{dz}{z^2 - iz}$$  \hspace{1cm} (47)
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\]

Example: $\mu_1 = 1/2$

\[
C(S, K, T) = Se^{-qT} - \frac{\sqrt{SK}}{\pi} e^{-\frac{(r+q)T}{2}} \int_0^{\infty} \text{Re} \left[ e^{-iu\kappa} \Phi \left( -u - \frac{i}{2} \right) \right] \frac{du}{u^2 + 1/4}
\]

where $\kappa = \ln(S/K) + (r - q)T$, $\Phi(u) = e^{iu\omega T} \Phi_X(u)$ and it is taken into account that the integrand is an even function of its real part.
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Example: $\mu_1 = 1/2$

$$C(S, K, T) = Se^{-qT} - \frac{\sqrt{SK}}{\pi} e^{-\frac{(r+q)T}{2}} \int_0^{\infty} \text{Re} \left[ e^{-iu\kappa} \Phi \left(-u - \frac{i}{2}\right) \right] \frac{du}{u^2 + 1/4}$$

where $\kappa = \ln(S/K) + (r - q)T$, $\Phi(u) = e^{iu\omega T} \phi_X(u)$ and it is taken into account that the integrand is an even function of its real part.
Lewis method - results

**Figure 10:** European option values in VG model at $T = 1.0yr$, $K = 90$, $\sigma = 0.1$ obtained with the new FFT method.

**Figure 11:** European option values in VG model at $T = 1.0yrs$, $K = 90$, $\sigma = 0.5$ obtained with the new FFT method.
**Lewis method - comparison**

Figure 12: European option values in VG model at $T = 1.0\, yr$, $K = 90$, $\sigma = 0.5$ obtained with the new FFT method (rotated graph).

Figure 13: The difference between the European call option values for the VG model obtained with Carr-Madan FFT method and the new FFT method. Parameters of the test are: $S = 100$, $T = 0.5\, yr$, $\sigma = 0.2$, $\nu = 0.1$, $\Theta = -0.33$, $r = q = 0$. at various strikes.

Black-Scholes-wise method
**Generalization of the Black-Scholes formula**


**Theorem:** Given $\phi_{X_t}(z)$ of the model $M$, price of an European option is

\[
\Pi^M_1 = \frac{1}{2} + \frac{\xi}{2\pi} \int_{-\infty}^{\infty} e^{-iu \ln K} e^{iu [\ln S + (r-q+\omega)T]} \phi_{X_T}(u-i) \frac{iu\phi_{X_T}(-i)}{i u \phi_{X_T}(-i)} du,
\]

\[
\Pi^M_2 = \frac{1}{2} + \frac{\xi}{2\pi} \int_{-\infty}^{\infty} e^{-iu \ln K} e^{iu [\ln S + (r-q+\omega)T]} \phi_{X_T}(u) du,
\]

\[
V^M = \xi \left[ e^{-qT} S_0 \Pi^M_1 - e^{-rT} K \Pi^M_2 \right],
\]

$\xi = 1(-1)$ for a call(put). By definition $\phi_{X_t}(0) = 1$, and $\phi_{X_t}(-i)$ is a function of $T$ and parameters of the model only.
Proof

Assume that $\phi_T( -z )$ has a strip of regularity $0 \leq \mu \leq 1$. Rewrite the Lewis formula as

$$C(S, K, T) = -\frac{Ke^{-rT}}{2\pi} \int_{i\mu - \infty}^{i\mu + \infty} e^{-izk} \phi_X( -z ) \frac{dz}{z^2 - iz} = -\frac{Ke^{-rT}}{2\pi} \left[ \right]$$

$$\int_{i\mu - \infty}^{i\mu + \infty} e^{-izk} \phi_X( -z ) \frac{idz}{z} - \int_{i\mu - \infty}^{i\mu + \infty} e^{-izk} \phi_X( -z ) \frac{idz}{z - i} = -\frac{Ke^{-rT}}{2\pi} (\mathcal{R}(I_1) - \mathcal{R}(I_2))$$
Proof

Assume that $\phi_T(-z)$ has a strip of regularity $0 \leq \mu \leq 1$. Rewrite the Lewis formula as

$$C(S, K, T) = -\frac{K e^{-rT}}{2\pi} \int_{i\mu - \infty}^{i\mu + \infty} e^{-izk} \phi_X(-z) \frac{dz}{z^2 - iz} = -\frac{K e^{-rT}}{2\pi} \left[ \right]$$

$$\int_{i\mu - \infty}^{i\mu + \infty} e^{-izk} \phi_X(-z) \frac{idz}{z} - \int_{i\mu - \infty}^{i\mu + \infty} e^{-izk} \phi_X(-z) \frac{idz}{z - i} \right] = -\frac{K e^{-rT}}{2\pi} (\Re(I_1) - \Re(I_2))$$

Contour integration and Cauchy theorem

Figure 15: Integration contour for $\Re(I_1)$

Proof - continue

\[ R(I_1) = \pi + \int_{-\infty}^{\infty} e^{-iu \ln K} e^{iu \ln S + (r-q+\omega)T} \frac{\phi_X(u)}{iu} \, du. \]

\[ R(I_2) = \frac{S}{K} e^{(r-q)T} \left( \pi + \int_{-\infty}^{\infty} e^{-iu \ln K} e^{iu \ln S + (r-q+\omega)T} \frac{\phi_X(u-i)}{i(u\phi_X(-i))} \, du \right). \]

The difficulty in using FFT to evaluate these integrals, as noted by Carr and Madan is the divergence of the integrands at \( u = 0 \). Specifically, let us develop the characteristic function \( \phi_X(z) \) with \( z = u + iv \) as Taylor series in \( u \)

\[ \phi_X(z) = \mathbb{E}[e^{-vX}] + iu \mathbb{E}[xe^{-vX}] - \frac{1}{2} u^2 \mathbb{E}[x^2e^{-vX}] + \ldots \quad (56) \]

We have to chose \( z = u - i \) for the first expression, and \( z = u \) in the second one. As it is easy to check in both cases that the leading term in the expansion under both integrals is \( 1/(iu) \) which is just a source of the divergence. The source of this divergence is a discontinuity of the payoff function at \( K = S_T \). Accordingly the Fourier transform of the payoff function has large high-frequency terms. The Carr-Madan solution is in fact to dampen the weight of the high frequencies by multiplying the payoff by an exponential decay function. This will lower the importance of the singularity, but at the cost of degradation of the solution accuracy.
As the generalized BS can be used whenever the characteristic function of the given model is known, we can apply it to the Black-Scholes model as well that gives us the Black-Scholes option price $V^{BS}$ which is a well known analytic expression.
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Now the idea is to rewrite representation of the option price in in the form

$$V^M = [V^M - V^{BS}] + V^{BS}. \quad (58)$$
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$$V^M = [V^M - V^{BS}] + V^{BS}. \quad (59)$$

The term in braces can now be computed with FFT as

$$\Pi_1^{M-BS} = \frac{\xi}{2\pi} \int_{-\infty}^{\infty} e^{-i\kappa} \left[ \phi_X(u-i) e^{i(u-i)\omega T} - \phi_{BS}(u-i) e^{-\frac{\sigma^2}{2}T} \right] \frac{1}{iu} du,$$

$$\Pi_2^{M-BS} = \frac{\xi}{2\pi} \int_{-\infty}^{\infty} e^{-i\kappa} \left[ \phi_X(u) e^{iu\omega T} - \phi_{BS}(u) \right] \frac{1}{iu} du,$$

$$V^M - V^{BS} = \xi \left[ e^{-qT} S_0 \Pi_1^{M-BS} - e^{-rT} K \Pi_2^{M-BS} \right],$$

where $\kappa = \ln(K/S) - (r - q)T$, $\phi_{BS}(u) = e^{-\frac{\sigma^2}{2}u^2}$ and $\phi_X(-i) = e^{-\omega T}$. This is possible because we have removed the divergence in the integrals.
In more detail, first terms of the nominator expansion in series on small $u$ are

\[
D_1|_{u=0} \equiv \phi_{X_t}(u)e^{iu\omega T} - \phi_{BS}(u) = T(\theta + \omega + \frac{\sigma^2}{2})iu + O(u^2)
\]

\[
D_2|_{u=0} \equiv \phi_{X_t}(u - i)e^{i(u - i)\omega T} - \phi_{BS}(u - i)e^{-\frac{\sigma^2}{2}T} = -\left(\sigma^2 + \frac{\theta + \sigma^2}{-1 + \nu(\theta + \sigma^2/2)} - \omega\right)iu + O(u^2)
\]
Figure 18: European option values in VG model. Difference between the CM and BS-wise solution with $D_{1,2}(u = 0)$ at $T = 1.0 \text{yr}$, $\sigma = 0.1$, $\theta = 0.1$, $\nu = 0.1$, $r = 5\%$, $q = 2\%$

Figure 19: European option values in VG model. Difference between the CM and BS-wise solution with $D_{1,2}(u = \epsilon)$ at $T = 1.0 \text{yr}$, $\sigma = 0.1$, $\theta = 0.1$, $\nu = 0.1$, $r = 5\%$, $q = 2\%$
Convergency and performance
A. Sepp reported that convergency of the Black-Scholes-wise method is approximately 3 times faster than that of the Lewis method. It could be understood since usage of the Black-Scholes-wise formula allows us to remove a part of the FFT error instead substituting it with the exact analytical solution of the Black-Scholes problem.
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Cont and Tankov also analyze the Lewis method. They emphasize the fact that the Lewis integral is much easier to approximate at infinity than that in the Carr-Madan method, because the integrand decays exponentially (due to the presence of characteristic function). However, the price to pay for this is having to choose $\mu_1$. This choice is a delicate issue because choosing big $\mu_1$ leads to slower decay rates at infinity and bigger truncation errors and when $\mu_1$ is close to one, the denominator diverges and the discretization error becomes large. For models with exponentially decaying tails of Levy measure, $\mu_1$ cannot be chosen a priori and must be adjusted depending on the model parameters.
Figure 20: Convergency of the Black-Scholes-wise method. Difference between the option price obtained with $N = 8192$, and that with $N = 4096, 1024, 512, 256$.

Figure 21: Convergency of the Lewis method. Difference between the option price obtained with $N = 8192$, and that with $N = 4096, 1024, 512, 256$. 

Figure 22: Convergency of the Carr-Madan method. Difference between the option price obtained with $N = 8192$, and that with $N = 4096, 1024, 512, 256$.

Figure 23: Convergency of all three methods.

Carr and Madan compare performance of 3 methods for computing VG prices for 160 strikes: VGP which is the analytic formula in Madan, Carr, and Chang; VGPS which computes delta and the risk-neutral probability of finishing in-the-money by Fourier inversion of the distribution function; VGFFTC which is a Carr-Madan method using FFT to invert the dampened call price; VGFFTIV which uses FFT to invert the modified time value.

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Table 2: CPU times for VG pricing (Carr-Madan 1999).
Performance - continue

Our calculations show that the performance of the Lewis method is same as the Carr-Madan method, and the performance of the Black-Scholes-wise method is only twice worse (because we need 2 FFT to compute 2 integrals).

<table>
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<td>0.078</td>
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</tbody>
</table>

Table 3: CPU times for VG pricing. Our calculations.
Conclusions
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- We showed that a popular Carr-Madan’s FFT method blows up for certain values of the model parameters even for European vanilla option.

- Alternative methods - one originally proposed by Lewis, and Black-Scholes-wise method were considered that seem to work fine for any value of the VG parameters.

- Convergency and accuracy of these methods is comparable with that of the Carr-Madan method, thus making them suitable for being used to price options with the VG model.
Thanks to Eugene and Dilip for VG!