Pricing options with VG model using FFT

Andrey Itkin

itkin@chem.ucla.edu

Moscow State Aviation University
Department of applied mathematics and physics

The idea of the work

- Discuss various analytic and numerical methods that have been used to price options within a framework of the VG model

The idea of the work

- Discuss various analytic and numerical methods that have been used to price options within a framework of the VG model
- Show that some popular methods, for instance, Carr-Madan's FFT method could blow up at certain values of the model parameters even for an European vanilla option

The idea of the work

- Discuss various analytic and numerical methods that have been used to price options within a framework of the VG model
- Show that some popular methods, for instance, Carr-Madan's FFT method could blow up at certain values of the model parameters even for an European vanilla option
- Consider alternative methods - Lewis-wise and Black-Scholes-wise and show that they seem to work fine for any value of the VG parameters

The idea of the work

- Discuss various analytic and numerical methods that have been used to price options within a framework of the VG model
- Show that some popular methods, for instance, Carr-Madan's FFT method could blow up at certain values of the model parameters even for an European vanilla option
- Consider alternative methods - Lewis-wise and Black-Scholes-wise and show that they seem to work fine for any value of the VG parameters
- Give test examples to demonstrate efficiency of these methods and discuss convergency and accuracy of all methods

Brief overview of the VG model

VG model

- Proposed by Madan and Seneta (1990) to describe stock price dynamics instead of the Brownian motion in the original Black-Scholes model.
- Two new parameters: θ skewness and ν kurtosis are introduced in order to describe asymmetry and fat tails of real life distributions.
- The VG process is defined by evaluating Brownian motion with drift at a random time specified by gamma process.

$$\ln F_t = \ln F_0 + X_t + \omega t, \tag{2}$$

where

$$X_t = \theta \gamma_t(1,\nu) + \sigma W_{\gamma_t(1,\nu)}, \tag{3}$$

VG model (continue)

and $\gamma_t(1,\nu)$ is a Gamma process playing the role of time in this case with unit mean rate and density function given by

$$f_{\gamma_t(1,\nu)}(x) = \frac{x^{\frac{t}{\nu}-1}e^{-\frac{x}{\nu}}}{\nu^{\frac{t}{\nu}}\Gamma\left(\frac{t}{\nu}\right)}. \tag{4}$$

The probability density function for the VG process may be written as

$$h_t(x) = \int_0^\infty \frac{dg}{\sqrt{2\pi g}} \exp\left[-\frac{(x-\theta g)^2}{2\sigma^2 g}\right] \frac{g^{\frac{t}{\nu}-1}e^{-\frac{g}{\nu}}}{\nu^{\frac{t}{\nu}}\Gamma\left(\frac{t}{\nu}\right)} \tag{5}$$

or after integration over g

$$h_t(x) = \frac{2e^{\theta x/\sigma^2}}{\sqrt{2}\pi\sigma\nu^{\frac{t}{\nu}}\Gamma(t/\nu)} \left(\frac{x^2}{\theta^2+\frac{2\sigma^2}{\nu}}\right)^{\frac{t}{2\nu}-\frac{1}{4}} K_{\frac{t}{\nu}-\frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{x^2\left(\theta^2+\frac{2\sigma^2}{\nu}\right)}\right), \tag{6}$$

VG model (continue)

where K is the modified Bessel function of the second kind. The characteristic function $\phi_{\gamma_t(1,\nu)}(u)$ for the VG process has a remarkably simple form

$$\phi_t(u) \equiv \langle E^{iux} \rangle \equiv \int_0^\infty h_t(x) e^{iux} dx = \frac{1}{(1 - i\theta\nu u + \frac{1}{2}\sigma^2\nu u^2)^{\frac{t}{\nu}}}. \qquad (7)$$

Now, to prevent arbitrage, we need F_t be a martingale, and, since F_t is already an independent increment process, all we need is

$$\mathbb{E}[F_t] = F_0, \qquad (8)$$

This tells us that

$$\omega = -\frac{\ln \phi_{X_t}(-i)}{t} = -\frac{-\frac{t}{\nu}\ln\left(1 - \theta\nu - \frac{1}{2}\sigma^2\nu\right)}{t} = \frac{1}{\nu}\ln\left(1 - \theta\nu - \frac{1}{2}\sigma^2\nu\right). \qquad (9)$$

VG model (continue)

From the definition of ω above, in order to have a risk neutral measure for VG model, its parameters must obey an inequality:

$$\frac{1}{\nu} > \theta + \frac{\sigma^2}{2}. \tag{10}$$

Accordingly, the characteristic function of the $x_T \equiv \log S_T$ VG process is

$$\phi(u) = \frac{\left[S_0 e^{(r-q+\omega)T}\right]^{iu}}{\left(1 - i\theta\nu u + \frac{1}{2}\sigma^2\nu u^2\right)^{\frac{T}{\nu}}}. \tag{11}$$

VG model (continue)

- Statistical parameters of VG distribution may be calculated from the historical data on stock prices.

VG model (continue)

- Statistical parameters of VG distribution may be calculated from the historical data on stock prices.
- In particular we have to find the values of the parameters θ^*, ν^* and σ^* such that the follwing expression is maximized:

VG model (continue)

- Statistical parameters of VG distribution may be calculated from the historical data on stock prices.
- In particular we have to find the values of the parameters θ^*, ν^* and σ^* such that the follwing expression is maximized:
- $$\prod_{j=1}^{n} h_{\tau_j}(x_j), \qquad (14)$$

  where the PDF $h_{\tau_j}(x_j)$ were given above, and x_j are observed returns per time τ_j, i.e. $x_j = \log(S_j/S_{j-1})$.

VG model (continue)

- Statistical parameters of VG distribution may be calculated from the historical data on stock prices.
- In particular we have to find the values of the parameters θ^*, ν^* and σ^* such that the follwing expression is maximized:
- $$\prod_{j=1}^{n} h_{\tau_j}(x_j), \tag{15}$$

  where the PDF $h_{\tau_j}(x_j)$ were given above, and x_j are observed returns per time τ_j, i.e. $x_j = \log(S_j/S_{j-1})$.
- Note that risk neutral parameters θ, ν, σ do not have to be equal to their statistical counterparts.

Pricing European option

European option value

The value of European option on a stock when the risk neutral dynamics is given by VG is

$$V = \exp(-rT) \int_{-\infty}^{\infty} h_T\left(x - (r - q + \omega)T\right) W(e^x) dx, \tag{16}$$

where T is time until expiration, q is continuous dividend and $W(e^x)$ is payoff function that has the following form

$$W(e^x) = (S_0 e^x - K)^+ - \text{call}, \quad W(e^x) = (K - S_0 e^x)^+ - \text{put}. \tag{17}$$

Direct calculation allows us to derive the put-call parity relation identical to Black-Scholes case

$$C = S_0 e^{-qT} - K e^{-rT} + P. \tag{18}$$

Existing methods of pricing

- Closed form solution derived in D. Madan, P. Carr, and E. Chang. "The variance gamma process and option pricing". *European Finance Review*, 2:79–105, 1998. - slow because it requires computation of modified Bessel functions.

Existing methods of pricing

- Closed form solution derived in D. Madan, P. Carr, and E. Chang. “The variance gamma process and option pricing”. *European Finance Review*, 2:79–105, 1998. - slow because it requires computation of modified Bessel functions.
- FFT methods. Carr and Madan method (1999) nowadays is almost standard in math finance. Can be used since the VG characteristic function has a very simple form given above. Unfortunately this method blows up at some values of the VG parameters as we will show below.

Existing methods of pricing

- Closed form solution derived in D. Madan, P. Carr, and E. Chang. "The variance gamma process and option pricing". *European Finance Review*, 2:79–105, 1998. - slow because it requires computation of modified Bessel functions.
- FFT methods. Carr and Madan method (1999) nowadays is almost standard in math finance. Can be used since the VG characteristic function has a very simple form given above. Unfortunately this method blows up at some values of the VG parameters as we will show below. FRFT!

Existing methods of pricing

- Closed form solution derived in D. Madan, P. Carr, and E. Chang. "The variance gamma process and option pricing". *European Finance Review*, 2:79–105, 1998. - slow because it requires computation of modified Bessel functions.
- FFT methods. Carr and Madan method (1999) nowadays is almost standard in math finance. Can be used since the VG characteristic function has a very simple form given above. Unfortunately this method blows up at some values of the VG parameters as we will show below. FRFT!
- Konikov and Madan in ("Variance gamma model: Gamma weighted Black-Scholes implementation", Technical report, Bloomberg L.P., 2004) This method leads to a weighted sum of the BS formulae while has not been implemented yet.

Existing methods of pricing

- Closed form solution derived in D. Madan, P. Carr, and E. Chang. "The variance gamma process and option pricing". *European Finance Review*, 2:79–105, 1998. - slow because it requires computation of modified Bessel functions.
- FFT methods. Carr and Madan method (1999) nowadays is almost standard in math finance. Can be used since the VG characteristic function has a very simple form given above. Unfortunately this method blows up at some values of the VG parameters as we will show below. FRFT!
- Konikov and Madan in ("Variance gamma model: Gamma weighted Black-Scholes implementation", Technical report, Bloomberg L.P., 2004) This method leads to a weighted sum of the BS formulae while has not been implemented yet.
- Monte-Carlo methods - slow.

Carr-Madan's FFT approach and VG

Once the characteristic function $\phi(u,t) = \mathbb{E}(e^{iuX_t})$, where $X_t = \log(S_t)$, is available, then the vanilla call option can be priced using Carr-Madan's FFT formula:

$$C(K,T) = \frac{e^{-\alpha \log(K)}}{\pi} \int_0^\infty \mathrm{Re}\left[e^{-iv\log(K)}\omega(v)\right] dv, \tag{19}$$

where

$$\omega(v) = \frac{e^{-rT}\phi(v-(\alpha+1)i, T)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v} \tag{20}$$

The integral in the first equation can be computed using FFT, and as a result we get call option prices for a variety of strikes. The put option values can just be constructed from Put-Call parity.

Parameter α must be positive. Usually $\alpha = 3$ works well for various models. It is important that the denominator has only imaginary roots while integration is provided along real v. Thus, the integrand $\omega(v)$ is well-behaved.

Carr-Madan's FFT approach and VG

Once the characteristic function $\phi(u,t) = \mathbb{E}(e^{iuX_t})$, where $X_t = \log(S_t)$, is available, then the vanilla call option can be priced using Carr-Madan's FFT formula:

$$C(K,T) = \frac{e^{-\alpha \log(K)}}{\pi} \int_0^\infty \mathrm{Re}\left[e^{-iv\log(K)}\omega(v)\right] dv, \tag{21}$$

where

$$\omega(v) = \frac{e^{-rT}\phi(v-(\alpha+1)i,T)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v} \tag{22}$$

The integral in the first equation can be computed using FFT, and as a result we get call option prices for a variety of strikes. The put option values can just be constructed from Put-Call parity.

Parameter α must be positive. Usually $\alpha = 3$ works well for various models. It is important that the denominator has only imaginary roots while integration is provided along real v. Thus, the integrand $\omega(v)$ is well-behaved.

Carr-Madan´s FFT approach and VG

Once the characteristic function $\phi(u,t) = \mathbb{E}(e^{iuX_t})$, where $X_t = \log(S_t)$, is available, then the vanilla call option can be priced using Carr-Madan´s FFT formula:

$$C(K,T) = \frac{e^{-\alpha \log(K)}}{\pi} \int_0^\infty \mathrm{Re}\left[e^{-iv\log(K)} \omega(v) \right] dv, \tag{23}$$

where

$$\omega(v) = \frac{e^{-rT}\phi(v-(\alpha+1)i,T)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v} \tag{24}$$

The integral in the first equation can be computed using FFT, and as a result we get call option prices for a variety of strikes. The put option values can just be constructed from Put-Call parity.

Parameter α must be positive. Usually $\alpha = 3$ works well for various models. It is important that the denominator has only imaginary roots while integration is provided along real v. Thus, the integrand $\omega(v)$ is well-behaved.

CM FFT results

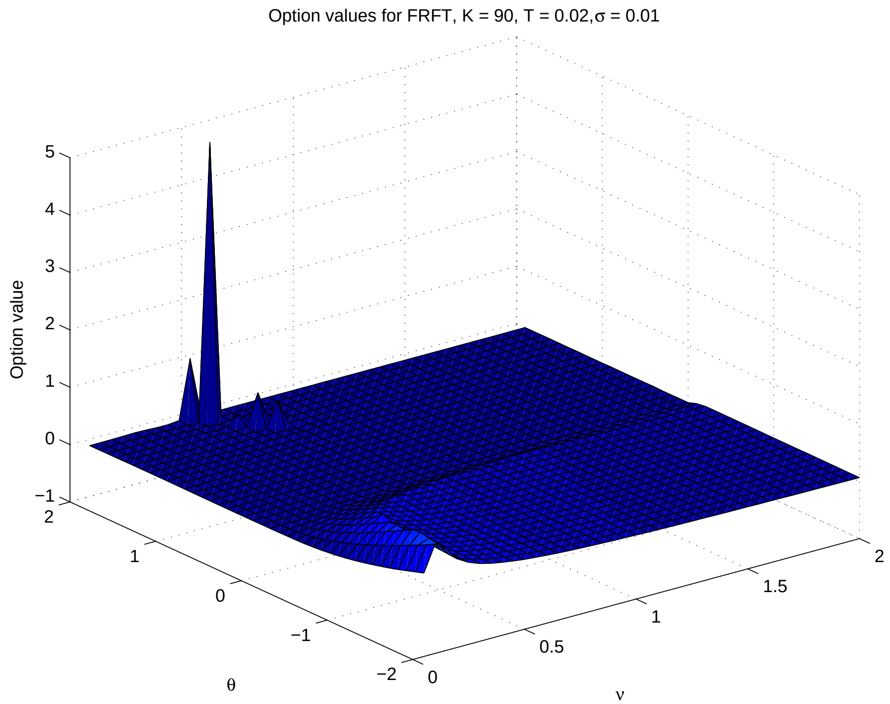


Figure 2: European option values in VG model at $T = 0.02yrs, K = 90, \sigma = 0.01$ obtained with FRFT.

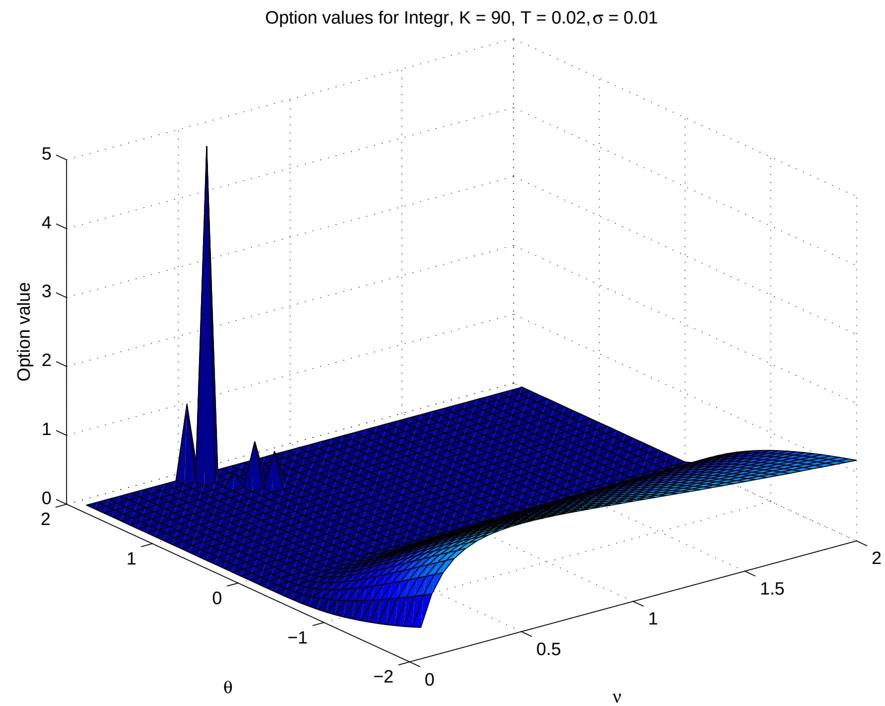


Figure 3: European option values in VG model at $T = 0.02yrs, K = 90, \sigma = 0.01$ obtained with the adaptive integration.

CM FFT results (continue)

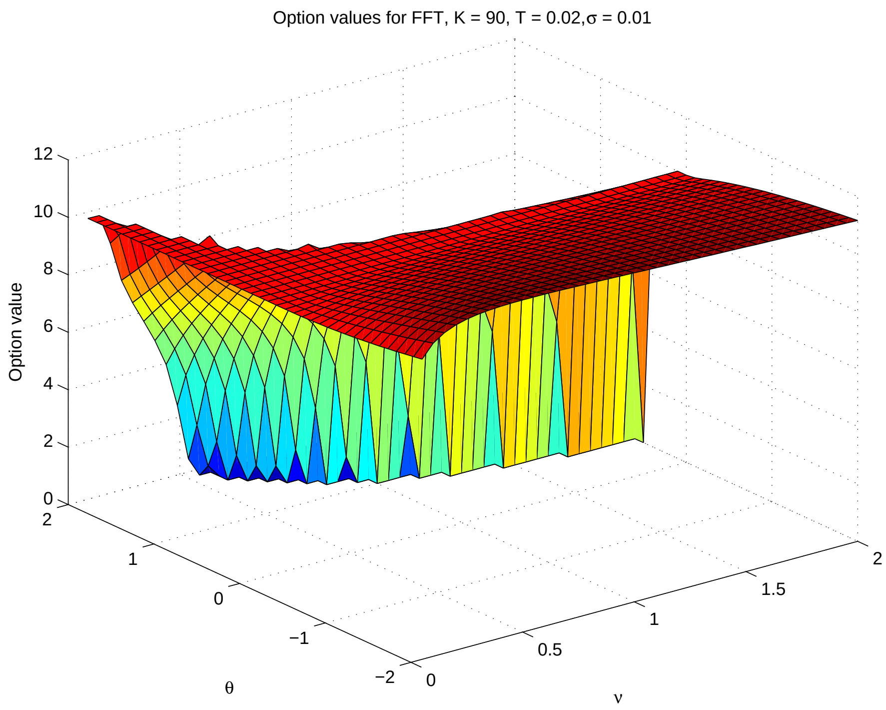


Figure 4: European option values in VG model at $T = 0.02yrs, K = 90, \sigma = 0.01$ obtained with FFT.

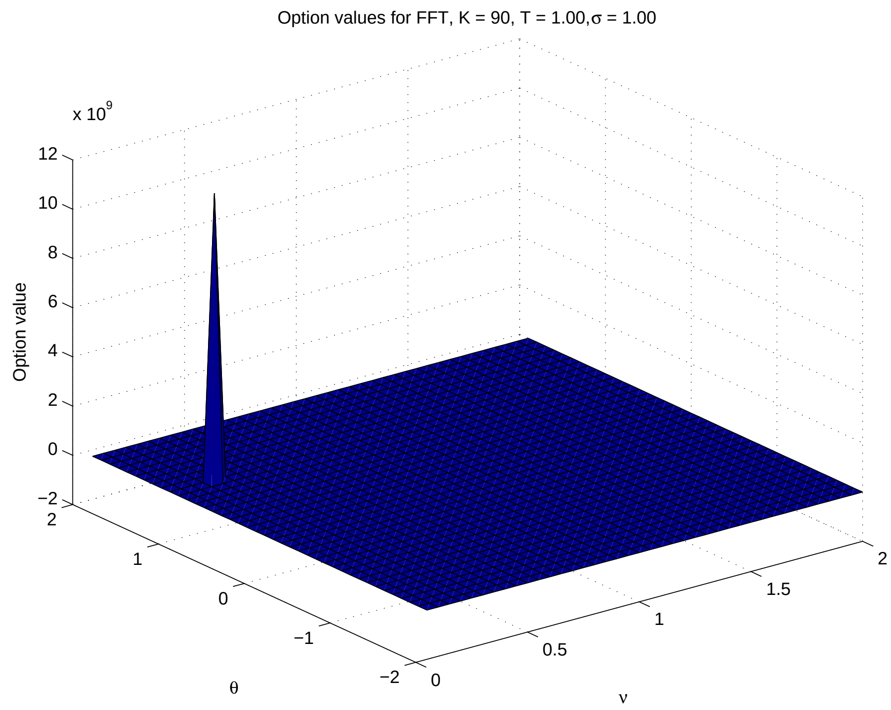


Figure 5: European option values in VG model at $T = 1.0yrs, K = 90, \sigma = 1.0$ obtained with the FFT.

CM FFT results (continue)

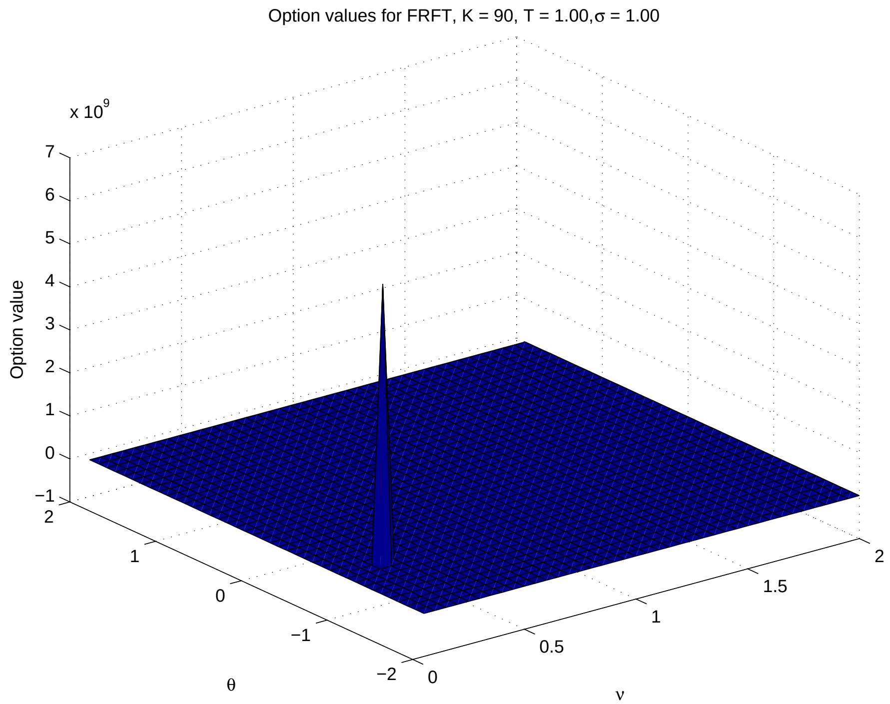


Figure 6: European option values in VG model at $T = 1.0yrs, K = 90, \sigma = 1.0$ obtained with the FRFT.

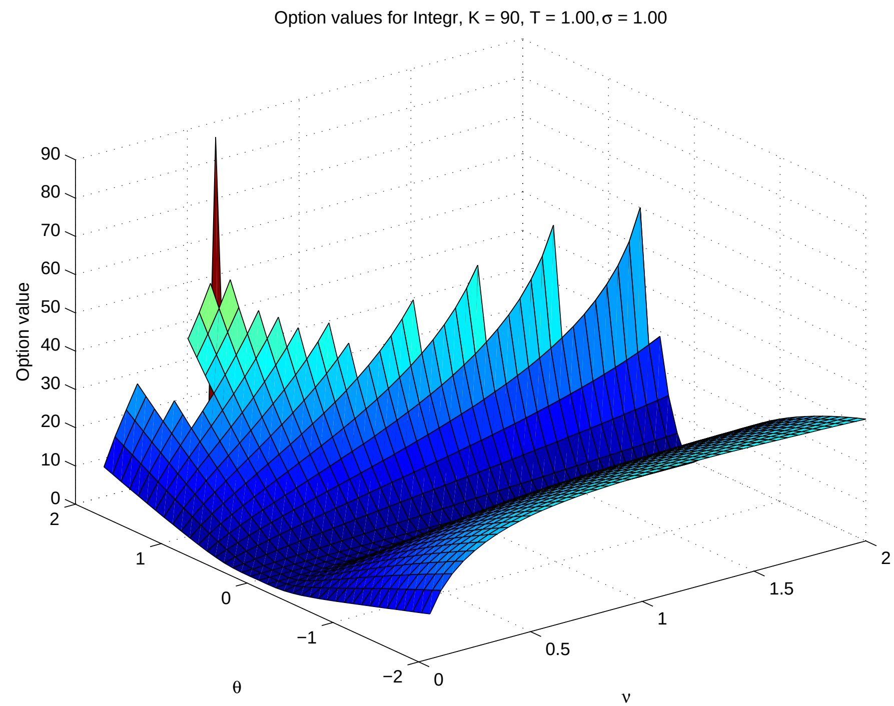


Figure 7: European option values in VG model at $T = 1.0yrs, K = 90, \sigma = 1.0$ obtained with the adaptive integration.

Investigation

European call option value in the Carr-Madan method

$$C(K,T) \propto \frac{e^{-\alpha \log(K) - rT}}{\pi} \int_0^\infty \Re(\Psi(v))dv \tag{25}$$

$$\Psi(v) \equiv \frac{e^{-iv\log(K)}}{\left[\alpha^2 + \alpha - v^2 + i(2\alpha+1)v\right]\left(1 - i\theta\nu u + \sigma^2\nu u^2/2\right)^{\frac{t}{\nu}}},$$

where $u \equiv v - (\alpha+1)i$. At small T the denominator has no real roots. To understand what happens at larger maturities, let us put $T = 0.8, \nu = 0.1, \alpha = 3, \sigma = 1$ and see how the denominator behaves as a function of v and Θ.

Investigation

European call option value in the Carr-Madan method

$$
\begin{aligned}
C(K,T) &\propto \frac{e^{-\alpha \log(K) - rT}}{\pi} \int_0^\infty \Re(\Psi(v))dv \qquad (27)\\
\Psi(v) &\equiv \frac{e^{-iv\log(K)}}{\left[\alpha^2 + \alpha - v^2 + i(2\alpha+1)v\right]\left(1 - i\theta\nu u + \sigma^2 \nu u^2/2\right)^{\frac{t}{\nu}}},
\end{aligned}
$$

where $u \equiv v - (\alpha+1)i$. At small T the denominator has no real roots. To understand what happens at larger maturities, let us put $T = 0.8, \nu = 0.1, \alpha = 3, \sigma = 1$ and see how the denominator behaves as a function of v and Θ.

Carr and Madan's condition to keep the characteristic function to be finite

$$
\alpha < \sqrt{\frac{2}{\nu\sigma^2} + \frac{\Theta^2}{\sigma^4}} - \frac{\Theta}{\sigma^2} - 1. \qquad (28)
$$

Investigation (continue)

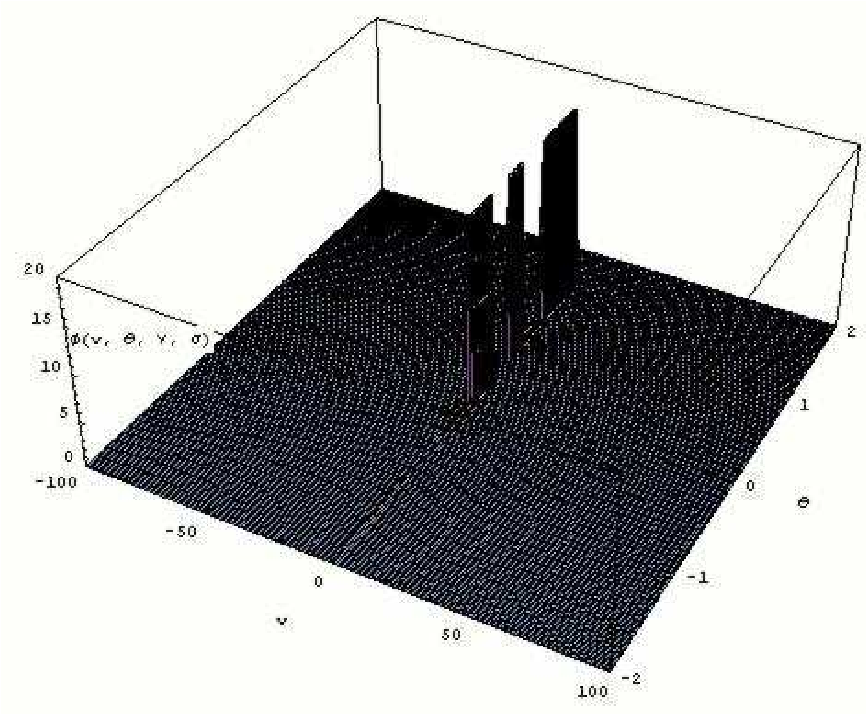


Figure 8: Denominator of $\Psi(v)$ at $T = 0.8, \nu = 0.1, \alpha = 3, \sigma = 1$ as a function of v and Θ.

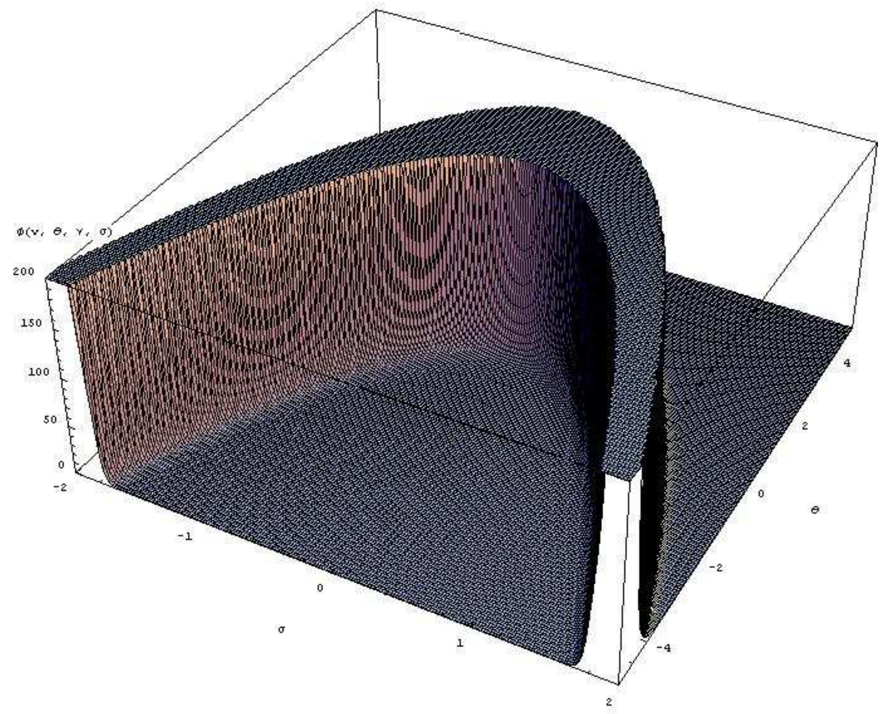


Figure 9: Denominator of $\Psi(v)$ at $T = 0.8, \nu = 0.1, \alpha = 3, v = 0$ as a function of σ and Θ.

Lewis regularization

Lewis method

Alan Lewis (2001) notes that a general integral representation of the European call option value with a vanilla payoff is

$$C_T(x_0, K) = e^{-rT} \int_{-\infty}^{\infty} (e^x - K)^+ Q(x, x_0, T) dx, \qquad (29)$$

where $x = \log S_T$ is a stock price that under a pricing measure evolves as $S_T = S_0 \exp[(r-q)T + X_T]$, and X_T is some Levy process satisfying $\mathbb{E}[exp(iuX_T)] = 1$, and Q is the density of the log-return distribution x.

The central point of the Lewis's work is to represent this equation as a convolution integral and then apply a Parseval identity

$$\int_{-\infty}^{\infty} f(x) g(x_0 - x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux_0} \hat{f}(u) \hat{g}(u) du, \qquad (30)$$

where the hat over function denotes its Fourier transform.

Lewis method (continue)

The idea behind this formula is that the Fourier transform of a transition probability density for a Levy process to reach $X_t = x$ after the elapse of time t is a well-known characteristic function. For Levy processes it is $\phi_t(u) = \mathbb{E}[\exp(iuX_t)], u \in \Re$, and typically has an analytic extension (a generalized Fourier transform) $u \to z \in \mathbb{C}$, regular in some strip $\mathcal{S}_X$ parallel to the real z-axis.

Lewis method (continue)

The idea behind this formula is that the Fourier transform of a transition probability density for a Levy process to reach $X_t = x$ after the elapse of time t is a well-known characteristic function. For Levy processes it is $\phi_t(u) = \mathbb{E}[\exp(iuX_t)], u \in \Re$, and typically has an analytic extension (a generalized Fourier transform) $u \to z \in \mathbb{C}$, regular in some strip $\mathcal{S}_X$ parallel to the real z-axis.

Suppose that the generalized Fourier transform of the payoff function $\hat{w}(z) = \int_{-\infty}^{\infty} e^{izx}(e^x - K)^+ dx$ and $\phi_t(z)$ both exist, the call option value is

$$
\begin{aligned}
C_T(x_0, K) &= e^{-rT}\mathbb{E}\left[(e^x - K)^+\right] = \frac{e^{-rT}}{2\pi}\mathbb{E}\left[\int_{i\mu-\infty}^{i\mu+\infty} e^{-izx_T}\hat{w}(z)dz\right] \\
&= \frac{e^{-rT}}{2\pi}\mathbb{E}\left[\int_{i\mu-\infty}^{i\mu+\infty} e^{-iz[x_0+(r-q+\omega)T]}e^{-izX_T}\hat{w}(z)dz\right] \\
&= \frac{e^{-rT}}{2\pi}\int_{i\mu-\infty}^{i\mu+\infty} e^{-izY}\phi_{X_T}(-z)\hat{w}(z)dz.
\end{aligned}
$$

Here $Y = x_0 + (r - q + \omega)T$, $\mu \equiv$ Im z. This is a formal derivation which becomes a valid proof if all the integrals exist.

Lewis method (continue)

The Fourier transform of the vanilla payoff can be easily found by a direct integration

$$\hat{w}(z) = \int_{-\infty}^{\infty} e^{izx}(e^x - K)^+ dx = -\frac{K^{iz+1}}{z^2 - iz}, \qquad \mathrm{Im} z > 1. \tag{33}$$

Lewis method (continue)

The Fourier transform of the vanilla payoff can be easily found by a direct integration

$$\hat{w}(z) = \int_{-\infty}^{\infty} e^{izx}(e^x - K)^+ dx = -\frac{K^{iz+1}}{z^2 - iz}, \qquad \mathrm{Im} z > 1. \tag{35}$$

Note that if z were real, this regular Fourier transform would not exist. As shown by Lewis, payoff transforms $\hat{w}(z)$ for typical claims exist and are regular in their own strips $\mathcal{S}_w$ in the complex z-plane, just like characteristic functions.

Lewis method (continue)

The Fourier transform of the vanilla payoff can be easily found by a direct integration

$$\hat{w}(z) = \int_{-\infty}^{\infty} e^{izx}(e^x - K)^+ dx = -\frac{K^{iz+1}}{z^2 - iz}, \qquad \mathrm{Im} z > 1. \tag{37}$$

Note that if z were real, this regular Fourier transform would not exist. As shown by Lewis, payoff transforms $\hat{w}(z)$ for typical claims exist and are regular in their own strips $\mathcal{S}_w$ in the complex z-plane, just like characteristic functions. Above we denoted the strip where the characteristic function $\phi(z)$ is well-behaved as $\mathcal{S}_X$. Therefore, $\phi(-z)$ is defined at the conjugate strip $\mathcal{S}_X^*$. Thus, the pricing formula is defined at the strip $\mathcal{S}_V = \mathcal{S}_X^* \bigcap \mathcal{S}_w$, where it has the form

$$C(S, K, T) = -\frac{Ke^{-rT}}{2\pi} \int_{i\mu-\infty}^{i\mu+\infty} e^{-izk} \phi_{X_T}(-z) \frac{dz}{z^2 - iz}, \quad \mu \in \mathcal{S}_V, \tag{38}$$

and $k = \log(S/K) + (r - q + \omega)T$.

Lewis method (continue)

The Fourier transform of the vanilla payoff can be easily found by a direct integration

$$\hat{w}(z) = \int_{-\infty}^{\infty} e^{izx}(e^x - K)^+ dx = -\frac{K^{iz+1}}{z^2 - iz}, \qquad \mathrm{Im} z > 1. \tag{39}$$

Note that if z were real, this regular Fourier transform would not exist. As shown by Lewis, payoff transforms $\hat{w}(z)$ for typical claims exist and are regular in their own strips $\mathcal{S}_w$ in the complex z-plane, just like characteristic functions. Above we denoted the strip where the characteristic function $\phi(z)$ is well-behaved as $\mathcal{S}_X$. Therefore, $\phi(-z)$ is defined at the conjugate strip $\mathcal{S}_X^*$. Thus, the pricing formula is defined at the strip $\mathcal{S}_V = \mathcal{S}_X^* \bigcap \mathcal{S}_w$, where it has the form

$$\boxed{C(S,K,T) = -\frac{Ke^{-rT}}{2\pi} \int_{i\mu-\infty}^{i\mu+\infty} e^{-izk} \phi_{X_T}(-z) \frac{dz}{z^2 - iz}, \qquad \mu \in \mathcal{S}_V,} \tag{40}$$

and $k = \log(S/K) + (r - q + \omega)T$.

Lewis method - existence

The characteristic function of the VG process is defined in the strip $\beta - \gamma < \text{Im } z < \beta + \gamma$, where

$$\beta = \frac{\Theta}{\sigma^2}, \quad \gamma = \sqrt{\frac{2}{\nu\sigma^2} + \frac{\Theta^2}{\sigma^4} + 2(\mathrm{Re} z)^2}. \tag{41}$$

This condition can be relaxed by assuming $\mathrm{Re} z = 0$. Accordingly, $\phi(-z)$ is defined in the strip $\gamma - \beta > \text{Im } z > -\beta - \gamma$.

Lewis method - existence

The characteristic function of the VG process is defined in the strip $\beta - \gamma < \text{Im } z < \beta + \gamma$, where

$$\beta = \frac{\Theta}{\sigma^2}, \quad \gamma = \sqrt{\frac{2}{\nu\sigma^2} + \frac{\Theta^2}{\sigma^4} + 2(\text{Re}z)^2}. \tag{43}$$

This condition can be relaxed by assuming $\text{Re}z = 0$. Accordingly, $\phi(-z)$ is defined in the strip $\gamma - \beta > \text{Im } z > -\beta - \gamma$. Choose Im z in the form

$$\mu \equiv \text{Im } z = \sqrt{1 + \frac{2\Theta}{\sigma^2} + \frac{\Theta^2}{\sigma^4}} - \frac{\Theta}{\sigma^2}. \tag{44}$$

It is easy to see that μ defined in such a way obeys the inequality $\mu < \gamma - \beta$. On the other hand, $\mu \geq 1$ at any value of Θ and positive volatilities σ, and the equality is reached when $\Theta = 0$. It means, that Im $z = \mu$ lies in the strip $\mathcal{S}_X^*$ as well as in the strip $\mathcal{S}_w$, i. e. $\mu \in \mathcal{S}_V = \mathcal{S}_X^* \bigcap \mathcal{S}_w$.

Lewis method - contour integration

- The integrand in Eq.(39) is regular throughout $\mathcal{S}_X^*$ except for simple poles at $z = 0$ and $z = i$. The pole at $z = 0$ has a residue $-Ke^{-rT}i/(2\pi)$, and the pole at $z = i$ has a residue $Se^{-qT}i/(2\pi)$

Lewis method - contour integration

- The integrand in Eq.(39) is regular throughout $\mathcal{S}_X^*$ except for simple poles at $z = 0$ and $z = i$. The pole at $z = 0$ has a residue $-Ke^{-rT}i/(2\pi)$, and the pole at $z = i$ has a residue $Se^{-qT}i/(2\pi)$
- Strip $\mathcal{S}_X^*$ is defined by the condition $\gamma - \beta > \mathrm{Im} z > -\beta - \gamma$, where $\gamma - \beta > 1$, and $-\beta - \gamma < 0$. We can move the integration contour to $\mu_1 \in (0, 1)$ and use the residue theorem.

Lewis method - contour integration

- The integrand in Eq.(39) is regular throughout $\mathcal{S}_X^*$ except for simple poles at $z = 0$ and $z = i$. The pole at $z = 0$ has a residue $-Ke^{-rT}i/(2\pi)$, and the pole at $z = i$ has a residue $Se^{-qT}i/(2\pi)$
- Strip $\mathcal{S}_X^*$ is defined by the condition $\gamma - \beta > \mathrm{Im} z > -\beta - \gamma$, where $\gamma - \beta > 1$, and $-\beta - \gamma < 0$. We can move the integration contour to $\mu_1 \in (0, 1)$ and use the residue theorem.
- First alternative formula

$$C(S,K,T) = Se^{-qT} - \frac{Ke^{-rT}}{2\pi} \int_{i\mu_1-\infty}^{i\mu_1+\infty} e^{-izk} \phi_{X_T}(-z) \frac{dz}{z^2 - iz} \quad (47)$$

Lewis method - contour integration

- The integrand in Eq.(39) is regular throughout $\mathcal{S}_X^*$ except for simple poles at $z = 0$ and $z = i$. The pole at $z = 0$ has a residue $-Ke^{-rT} i/(2\pi)$, and the pole at $z = i$ has a residue $Se^{-qT} i/(2\pi)$
- Strip $\mathcal{S}_X^*$ is defined by the condition $\gamma - \beta > \mathrm{Im} z > -\beta - \gamma$, where $\gamma - \beta > 1$, and $-\beta - \gamma < 0$. We can move the integration contour to $\mu_1 \in (0, 1)$ and use the residue theorem.
- First alternative formula

$$C(S, K, T) = Se^{-qT} - \frac{Ke^{-rT}}{2\pi} \int_{i\mu_1 - \infty}^{i\mu_1 + \infty} e^{-izk} \phi_{X_T}(-z) \frac{dz}{z^2 - iz} \quad (48)$$

Example: $\mu_1 = 1/2$

$$\boxed{C(S, K, T) = Se^{-qT} - \frac{\sqrt{SK}}{\pi} e^{-\frac{(r+q)T}{2}} \int_0^\infty \mathrm{Re}\left[e^{-iu\kappa} \Phi\left(-u - \frac{i}{2}\right)\right] \frac{du}{u^2 + 1/4}}$$

where $\kappa = \ln(S/K) + (r - q)T, \quad \Phi(u) = e^{iu\omega T} \phi_{X_T}(u)$ and it is taken into account that the integrand is an even function of its real part.

Lewis method - contour integration

- The integrand in Eq.(39) is regular throughout $\mathcal{S}_X^*$ except for simple poles at $z = 0$ and $z = i$. The pole at $z = 0$ has a residue $-Ke^{-rT} i/(2\pi)$, and the pole at $z = i$ has a residue $Se^{-qT} i/(2\pi)$
- Strip $\mathcal{S}_X^*$ is defined by the condition $\gamma - \beta > \mathrm{Im} z > -\beta - \gamma$, where $\gamma - \beta > 1$, and $-\beta - \gamma < 0$. We can move the integration contour to $\mu_1 \in (0, 1)$ and use the residue theorem.
- First alternative formula

$$C(S,K,T) = Se^{-qT} - \frac{Ke^{-rT}}{2\pi} \int_{i\mu_1-\infty}^{i\mu_1+\infty} e^{-izk} \phi_{X_T}(-z) \frac{dz}{z^2 - iz} \quad (50)$$

Example: $\mu_1 = 1/2$

$$\boxed{C(S,K,T) = Se^{-qT} - \frac{\sqrt{SK}}{\pi} e^{-\frac{(r+q)T}{2}} \int_0^\infty \mathrm{Re}\left[e^{-iu\kappa} \ \Phi\left(-u - \frac{i}{2}\right)\right] \frac{du}{u^2 + 1/4}}$$

where $\kappa = \ln(S/K) + (r-q)T, \quad \Phi(u) = e^{iu\omega T} \phi_{X_T}(u)$ and it is taken into account that the integrand is an even function of its real part.

Lewis method - results

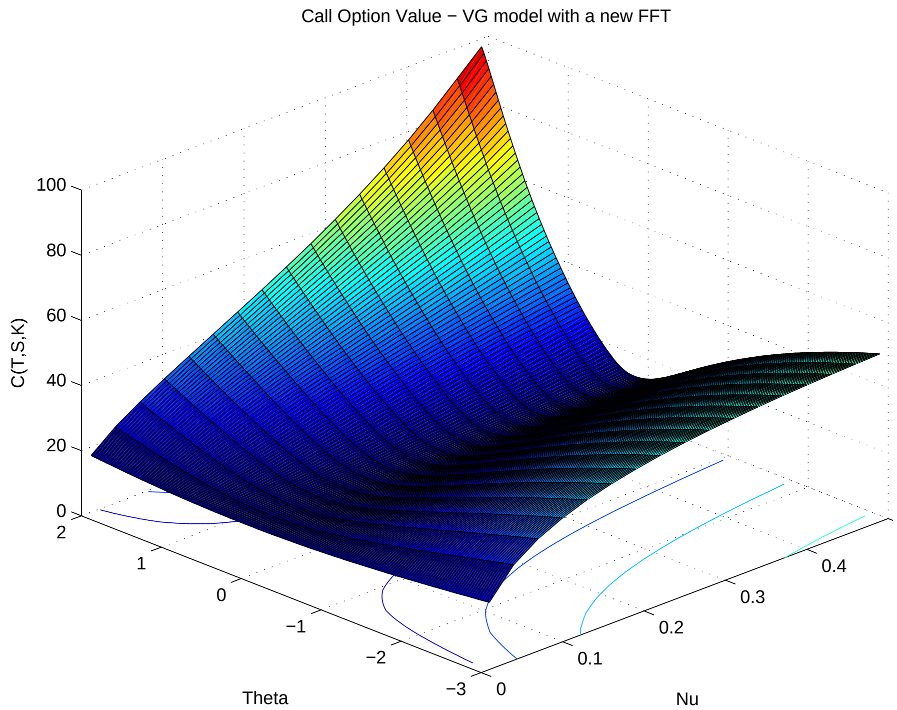


Figure 10: European option values in VG model at $T = 1.0yr, K = 90, \sigma = 0.1$ obtained with the new FFT method.

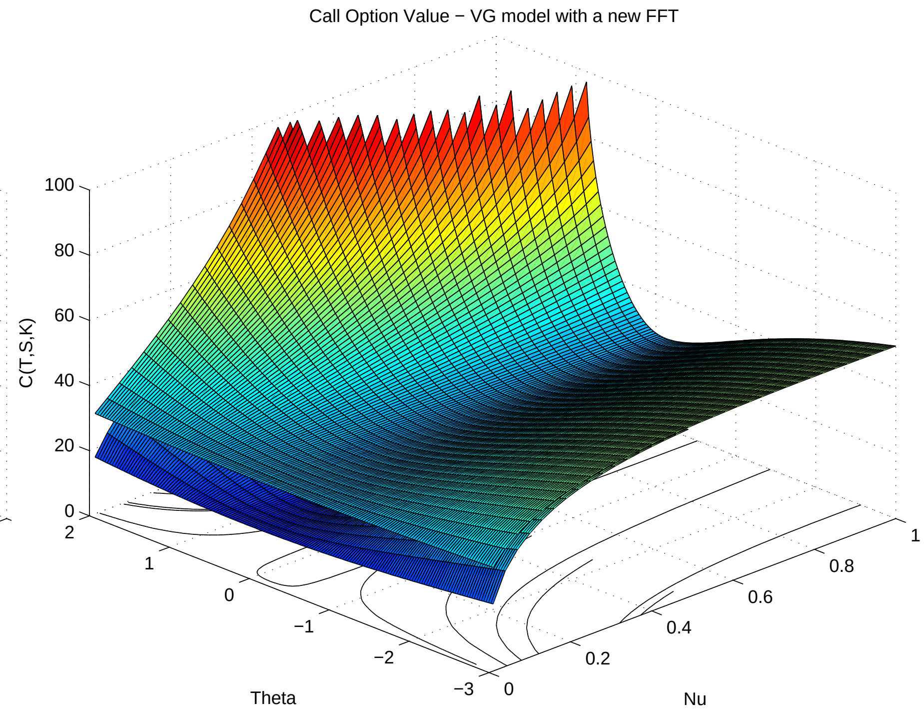


Figure 11: European option values in VG model at $T = 1.0yrs, K = 90, \sigma = 0.5$ obtained with the new FFT method.

Lewis method - comparison

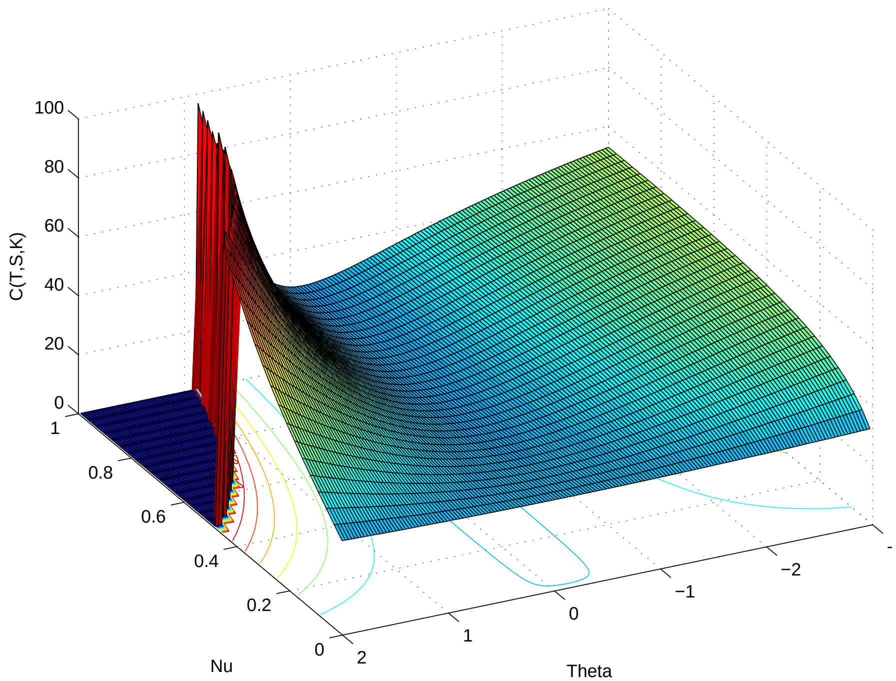


Figure 12: European option values in VG model at $T = 1.0yr, K = 90, \sigma = 0.5$ obtained with the new FFT method (rotated graph).

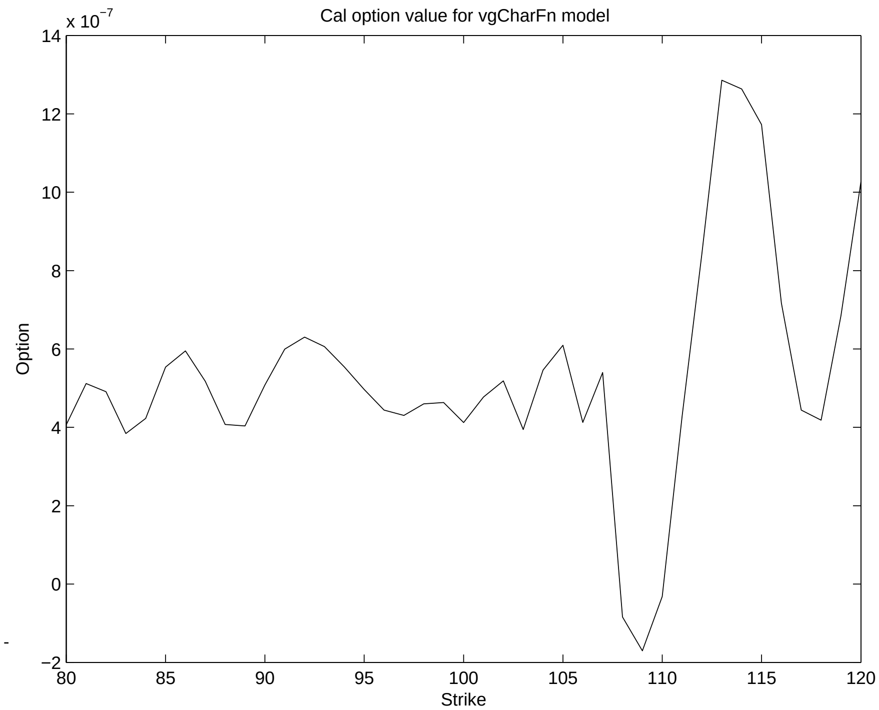


Figure 13: The difference between the European call option values for the VG model obtained with Carr-Madan FFT method and the new FFT method. Parameters of the test are: $S = 100, T = 0.5yr, \sigma = 0.2, \nu = 0.1, \Theta = -0.33, r = q = 0.$ at various strikes).

Black-Scholes-wise method

Generalization of the Black-Scholes formula

- General idea is discussed by A. Sepp ("Fourier transform for option pricing under affine-jump-diffusions: An overview", *Unpublished Manuscript, available at www.hot.ee/seppar*, 2003.), I. Yekutieli ("Pricing European options with FFT", Technical report, Bloomberg L.P., November 2004), and R. Cont and P. Tankov (*Financial modelling with jump processes*. Chapman & Hall / CRC, 2004.).

Generalization of the Black-Scholes formula

- General idea is discussed by A. Sepp ("Fourier transform for option pricing under affine-jump-diffusions: An overview", *Unpublished Manuscript, available at www.hot.ee/seppar*, 2003.), I. Yekutieli ("Pricing European options with FFT", Technical report, Bloomberg L.P., November 2004), and R. Cont and P. Tankov (*Financial modelling with jump processes*. Chapman & Hall / CRC, 2004.).

- **Theorem:** Given $\phi_{X_t}(z)$ of the model M, price of an European option is

$$
\begin{aligned}
\Pi_1^M &= \frac{1}{2} + \frac{\xi}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-iu\ln K} e^{iu[\ln S + (r-q+\omega)T]}\phi_{X_T}(u-i)}{iu\phi_{X_T}(-i)} du, \\
\Pi_2^M &= \frac{1}{2} + \frac{\xi}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-iu\ln K} e^{iu[\ln S + (r-q+\omega)T]}\phi_{X_T}(u)}{iu} du, \\
V^M &= \xi\left[e^{-qT} S_0 \Pi_1^M - e^{-rT} K \Pi_2^M\right],
\end{aligned}
$$

$\xi = 1(-1)$ for a call(put). By definition $\phi_{X_t}(0) = 1$, and $\phi_{X_t}(-i)$ is a function of T and parameters of the model only.

Proof

- Assume that $\phi_T(-z)$ has a strip of regularity $0 \le \mu \le 1$. Rewrite the Lewis formula as

$$C(S,K,T) = -\frac{Ke^{-rT}}{2\pi}\int_{i\mu-\infty}^{i\mu+\infty} e^{-izk}\phi_{X_T}(-z)\frac{dz}{z^2-iz} = -\frac{Ke^{-rT}}{2\pi}\Bigg[$$

$$\int_{i\mu-\infty}^{i\mu+\infty} e^{-izk}\phi_{X_T}(-z)\frac{idz}{z} - \int_{i\mu-\infty}^{i\mu+\infty} e^{-izk}\phi_{X_T}(-z)\frac{idz}{z-i}\Bigg] = -\frac{Ke^{-rT}}{2\pi}(\mathcal{R}(I_1) - \mathcal{R}(I_2))$$

Proof

- Assume that $\phi_T(-z)$ has a strip of regularity $0 \le \mu \le 1$. Rewrite the Lewis formula as

$$C(S,K,T) = -\frac{Ke^{-rT}}{2\pi}\int_{i\mu-\infty}^{i\mu+\infty} e^{-izk}\phi_{X_T}(-z)\frac{dz}{z^2-iz} = -\frac{Ke^{-rT}}{2\pi}\Big[$$

$$\int_{i\mu-\infty}^{i\mu+\infty} e^{-izk}\phi_{X_T}(-z)\frac{idz}{z} - \int_{i\mu-\infty}^{i\mu+\infty} e^{-izk}\phi_{X_T}(-z)\frac{idz}{z-i}\Big] = -\frac{Ke^{-rT}}{2\pi}(\mathcal{R}(I_1)-\mathcal{R}(I_2))$$

- Contour integration and Cauchy theorem

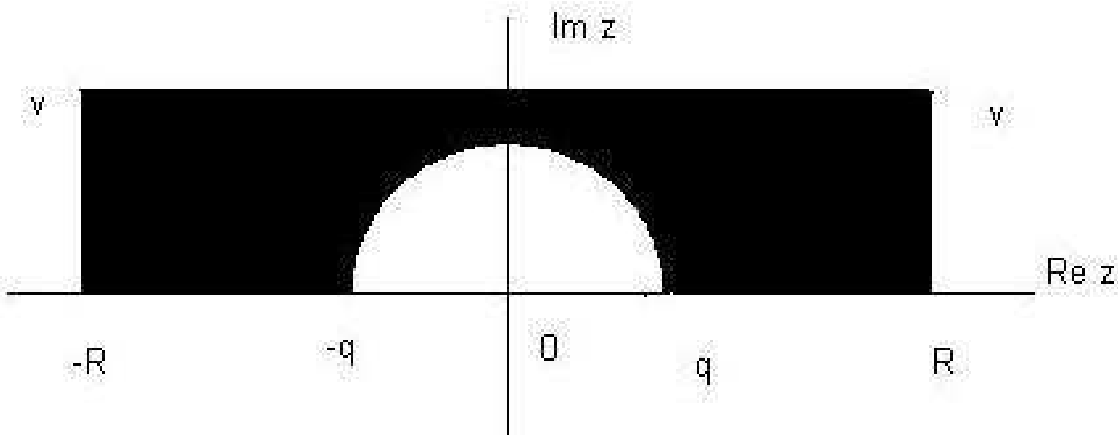


Figure 15: Integration contour for $\mathcal{R}(I_1)$

Proof - continue

$$\mathcal{R}(I_1) = \pi + \int_{-\infty}^{\infty} e^{-iu\ln K} e^{iu[\ln S + (r-q+\omega)T]} \frac{\phi_{X_T}(u)}{iu} du.$$

$$\mathcal{R}(I_2) = \frac{S}{K} e^{(r-q)T} \left(\pi + \int_{-\infty}^{\infty} e^{-iu\ln K} e^{iu[\ln S + (r-q+\omega)T]} \frac{\phi_{X_T}(u-i)}{iu\phi_{X_T}(-i)} du \right).$$

The difficulty in using FFT to evaluate these integrals, as noted by Carr and Madan is the divergence of the integrands at $u = 0$. Specifically, let us develop the characteristic function $\phi_{X_t}(z)$ with $z = u + iv$ as Taylor series in u

$$\phi_{X_t}(z) = \mathbb{E}[e^{-vX_t}] + iu\mathbb{E}[xe^{-vX_t}] - \frac{1}{2}u^2\mathbb{E}[x^2 e^{-vX_t}] + \dots \tag{56}$$

We have to chose $z = u - i$ for the first expression, and $z = u$ in the second one. As it is easy to check in both cases that the leading term in the expansion under both integrals is $1/(iu)$ which is just a source of the divergence.The source of this divergence is a discontinuity of the payoff function at $K = S_T$. Accordingly the Fourier transform of the payoff function has large high-frequency terms. The Carr-Madan solution is in fact to dampen the weight of the high frequencies by multiplying the payoff by an exponential decay function. This will lower the importance of the singularity, but at the cost of degradation of the solution accuracy.

New Idea - I. Yekutieli, Cont & Tankov

- As the generalized BS can be used whenever the characteristic function of the given model is known, we can apply it to the Black-Scholes model as well that gives us the Black-Scholes option price V^{BS} which is a well known analytic expression.

New Idea - I. Yekutieli, Cont & Tankov

- As the generalized BS can be used whenever the characteristic function of the given model is known, we can apply it to the Black-Scholes model as well that gives us the Black-Scholes option price V^{BS} which is a well known analytic expression.

- Now the idea is to rewrite representation of the option price in in the form

$$V^M = [V^M - V^{BS}] + V^{BS}. \tag{58}$$

New Idea - I. Yekutieli, Cont & Tankov

- As the generalized BS can be used whenever the characteristic function of the given model is known, we can apply it to the Black-Scholes model as well that gives us the Black-Scholes option price V^{BS} which is a well known analytic expression.

- Now the idea is to rewrite representation of the option price in in the form

$$V^M = [V^M - V^{BS}] + V^{BS}. \tag{59}$$

- The term in braces can now be computed with FFT as

$$\Pi_1^{M-BS} = \frac{\xi}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-iu\kappa}\left[\phi_{X_t}(u-i)e^{i(u-i)\omega T} - \phi_{BS}(u-i)e^{-\frac{\sigma^2}{2}T}\right]}{iu} du,$$

$$\Pi_2^{M-BS} = \frac{\xi}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-iu\kappa}\left[\phi_{X_t}(u)e^{iu\omega T} - \phi_{BS}(u)\right]}{iu} du,$$

$$V^M - V^{BS} = \xi\left[e^{-qT} S_0 \Pi_1^{M-BS} - e^{-rT} K \Pi_2^{M-BS}\right],$$

where $\kappa = \ln(K/S) - (r-q)T$, $\phi_{BS}(u) = e^{-\frac{\sigma^2 T}{2}u^2}$ and $\phi_{X_T}(-i) = e^{-\omega T}$. This is possible because we have removed the divergence in the integrals.

BS - results

In more detail, first terms of the nominator expansion in series on small u are

$$
\begin{aligned}
D_1|_{u=0} &\equiv \phi_{X_t}(u)e^{iu\omega T} - \phi_{BS}(u) = T(\theta + \omega + \frac{\sigma^2}{2})iu + O(u^2) \\
D_2|_{u=0} &\equiv \phi_{X_t}(u-i)e^{i(u-i)\omega T} - \phi_{BS}(u-i)e^{-\frac{\sigma^2}{2}T} \\
&= -\left(\sigma^2 + \frac{\theta+\sigma^2}{-1+\nu(\theta+\sigma^2/2)} - \omega\right) iu + O(u^2)
\end{aligned}
$$

BS - results

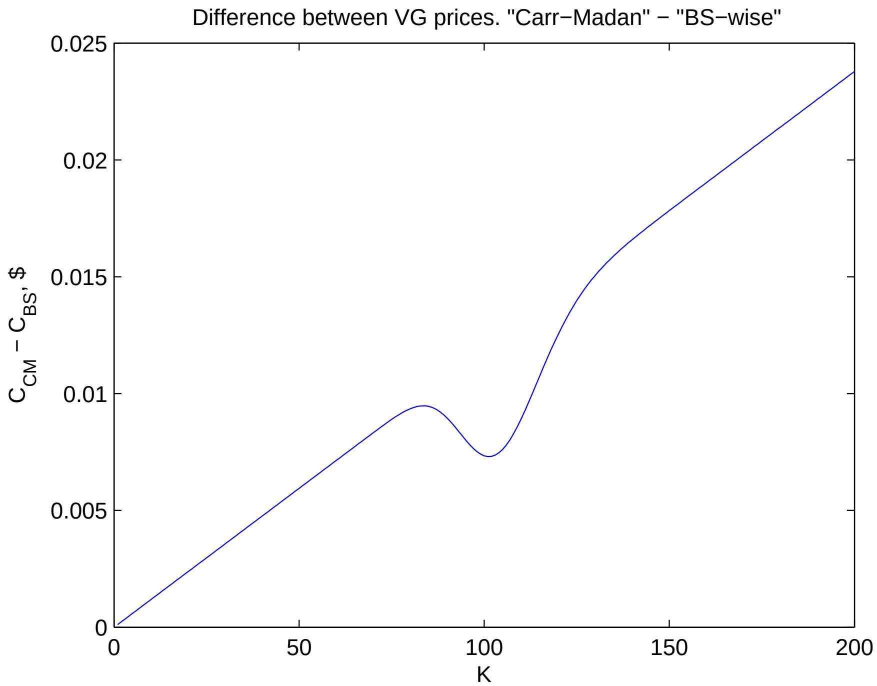


Figure 18: European option values in VG model. Difference between the CM and BS-wise solution with $D_{1,2}(u = 0)$ at $T = 1.0yr, \sigma = 0.1, \theta = 0.1, \nu = 0.1, r = 5\%, q = 2\%$

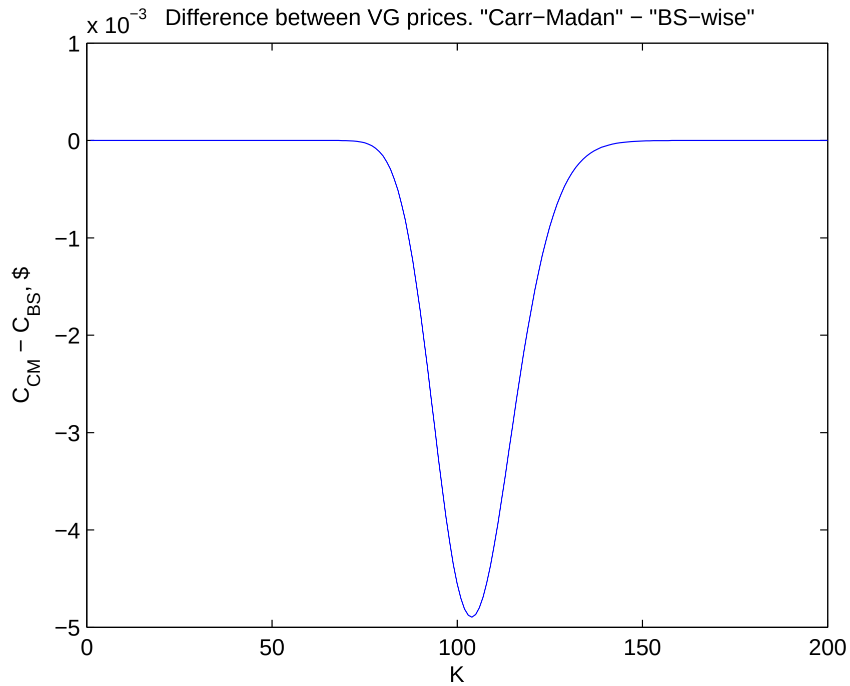


Figure 19: European option values in VG model. Difference between the CM and BS-wise solution with $D_{1,2}(u = \epsilon)$ at $T = 1.0yr, \sigma = 0.1, \theta = 0.1, \nu = 0.1, r = 5\%, q = 2\%$

Convergency and performance

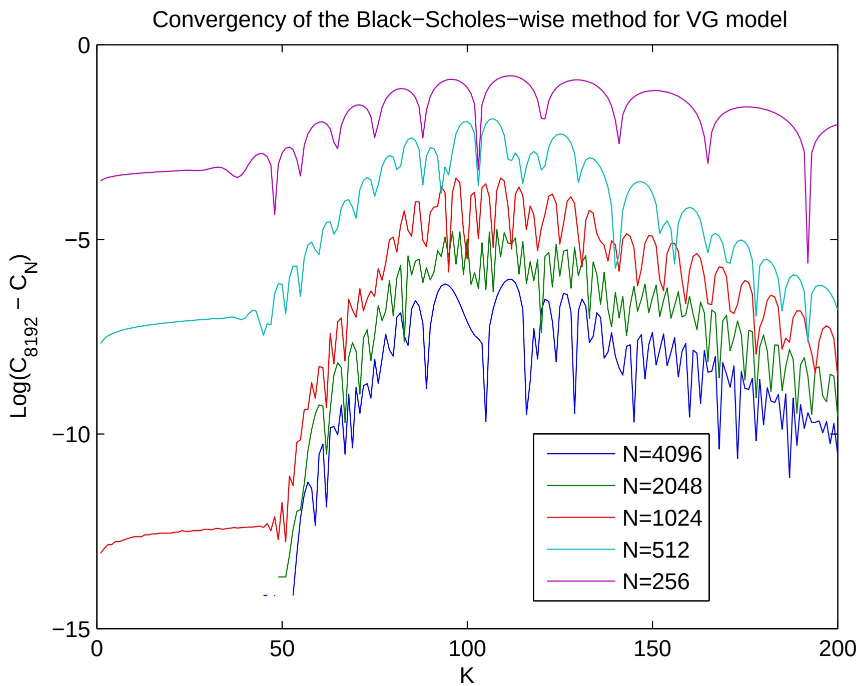
Convergency

- A. Sepp reported that convergency of the Black-Scholes-wise method is approximately 3 times faster than that of the Lewis method. It could be understood since usage of the Black-Scholes-wise formula allows us to remove a part of the FFT error instead substituting it with the exact analytical solution of the Black-Scholes problem.

Convergency

- A. Sepp reported that convergency of the Black-Scholes-wise method is approximately 3 times faster than that of the Lewis method. It could be understood since usage of the Black-Scholes-wise formula allows us to remove a part of the FFT error instead substituting it with the exact analytical solution of the Black-Scholes problem.
- Cont and Tankov also analyze the Lewis method. They emphasize the fact that the Lewis integral is much easier to approximate at infinity than that in the Carr-Madan method, because the integrand decays exponentially (due to the presence of characteristic function). However, the price to pay for this is having to choose μ_1. This choice is a delicate issue because choosing big μ_1 leads to slower decay rates at infinity and bigger truncation errors and when μ_1 is close to one, the denominator diverges and the discretization error becomes large. For models with exponentially decaying tails of Levy measure, μ_1 cannot be chosen a priori and must be adjusted depending on the model parameters.

Convergency - results



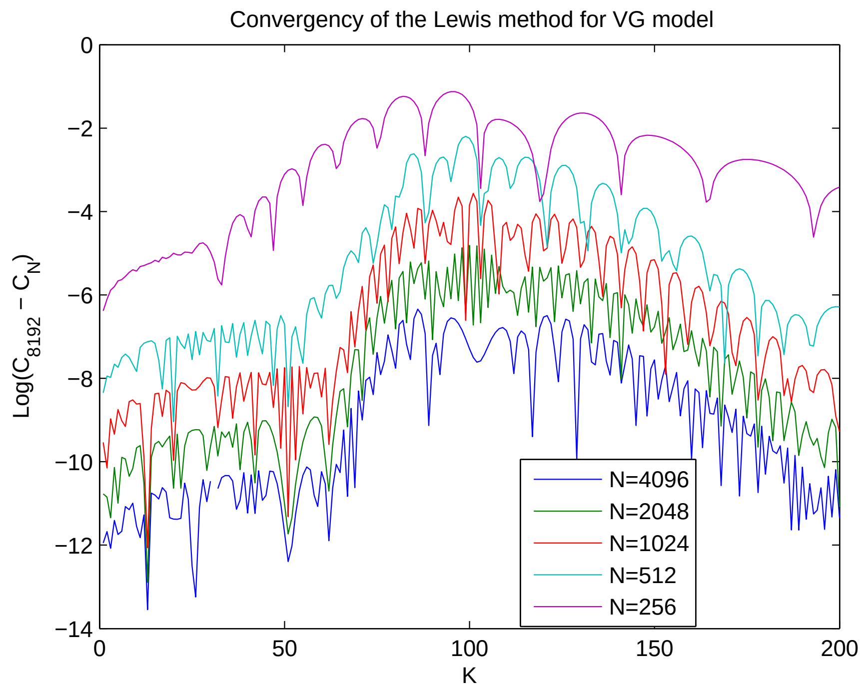


Figure 20: Convergency of the Black-Scholes-wise method. Difference between the option price obtained with $N = 8192$, and that with $N = 4096, 1024, 512, 256$

.

Figure 21: Convergency of the Lewis method. Difference between the option price obtained with $N = 8192$, and that with $N = 4096, 1024, 512, 256$

.

Convergency - results

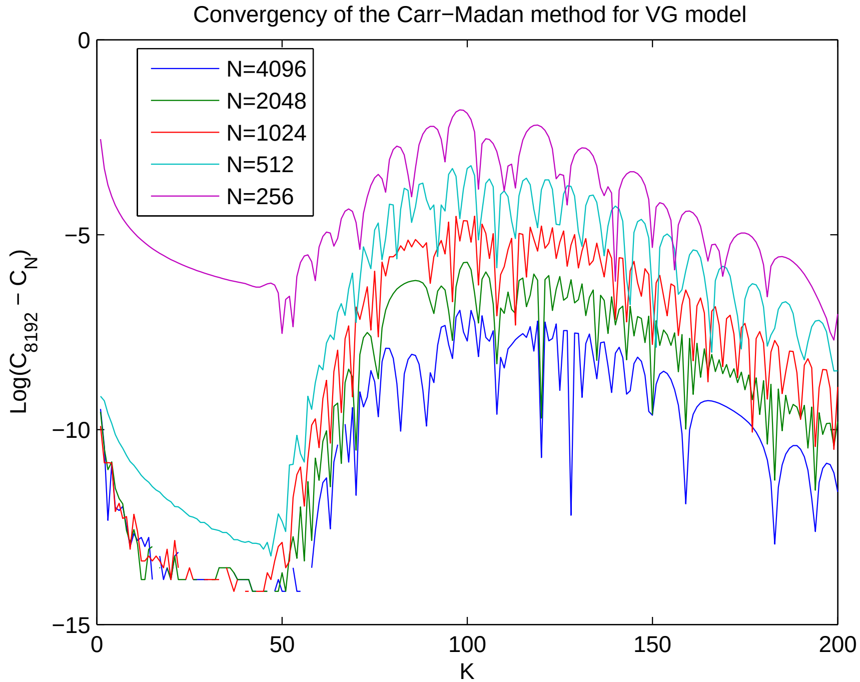


Figure 22: Convergency of the Carr-Madan method. Difference between the option price obtained with $N = 8192$, and that with $N = 4096, 1024, 512, 256$

.

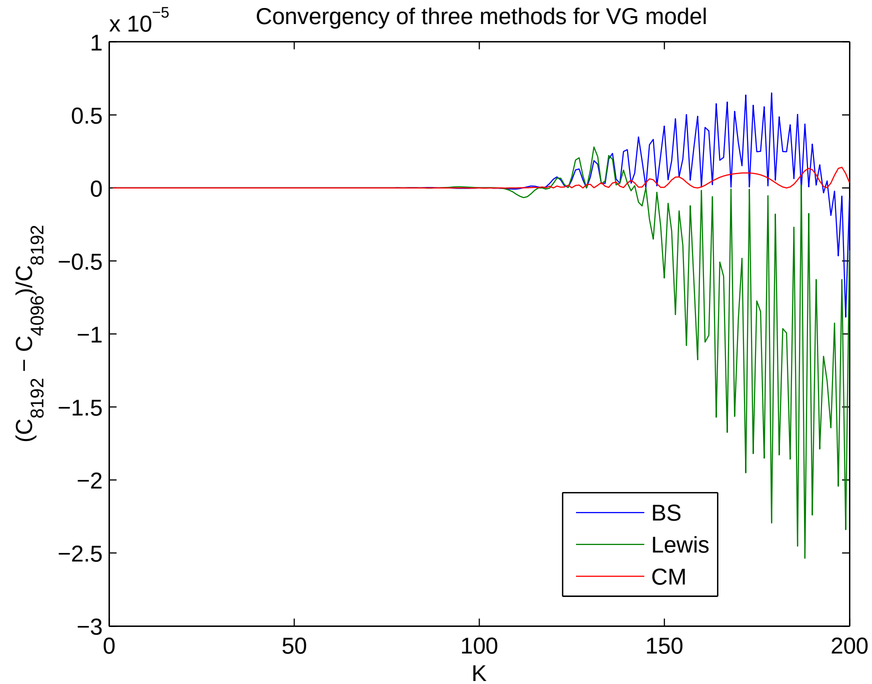


Figure 23: Convergency of all three methods

.

Performance

Carr and Madan compare performance of 3 methods for computing VG prices fro 160 strikes: VGP which is the analytic formula in Madan, Carr, and Chang; VGPS which computes delta and the risk-neutral probability of finishing in-the-money by Fourier inversion of the distribution function; VGFFTC which is a Carr-Madan method using FFT to invert the dampened call price; VGFFTTV which uses FFT to invert the modified time value.

	case 1	case 2	case 3	case 4
σ	.12	.25	.12	.25
ν	.16	2.0	.16	2.0
θ	-.33	-.10	-.33	-.10
T	1	1	.25	.25
VGP	22.41	24.81	23.82	24.74
VGPS	288.50	191.06	181.62	197.97
VGFFTC	6.09	6.48	6.72	6.52
VGFFTTV	11.53	11.48	11.57	11.56

Table 2: CPU times for VG pricing (Carr-Madan 1999).

Performance - continue

Our calculations show that the performance of the Lewis method is same as the Carr-Madan method, and the performance of the Black-Scholes-wise method is only twice worse (because we need 2 FFT to compute 2 integrals).

	case 1	case 2	case 3	case 4
σ	.12	.25	.12	.25
ν	.16	2.0	.16	2.0
θ	-.33	-.10	-.33	-.10
T	1	1	.25	.25
Lewis	0.031	0.031	0.031	0.031
Carr-Madan	0.047	0.047	0.032	0.032
BS-wise	0.078	0.078	0.062	0.062

Table 3: CPU times for VG pricing. Our calculations.

Conclusions

Conclusions

- We discussed various analytic and numerical methods that have been used to get option prices within a framework of VG model.

Conclusions

- We discussed various analytic and numerical methods that have been used to get option prices within a framework of VG model.
- We showed that a popular Carr-Madan's FFT method blows up for certain values of the model parameters even for European vanilla option.

Conclusions

- We discussed various analytic and numerical methods that have been used to get option prices within a framework of VG model.
- We showed that a popular Carr-Madan's FFT method blows up for certain values of the model parameters even for European vanilla option.
- Alternative methods - one originally proposed by Lewis, and Black-Scholes-wise method were considered that seem to work fine for any value of the VG parameters.

Conclusions

- We discussed various analytic and numerical methods that have been used to get option prices within a framework of VG model.
- We showed that a popular Carr-Madan's FFT method blows up for certain values of the model parameters even for European vanilla option.
- Alternative methods - one originally proposed by Lewis, and Black-Scholes-wise method were considered that seem to work fine for any value of the VG parameters.
- Convergency and accuracy of these methods is comparable with that of the Carr-Madan method, thus making them suitable for being used to price options with the VG model.

Thanks to Eugene and Dilip for VG!